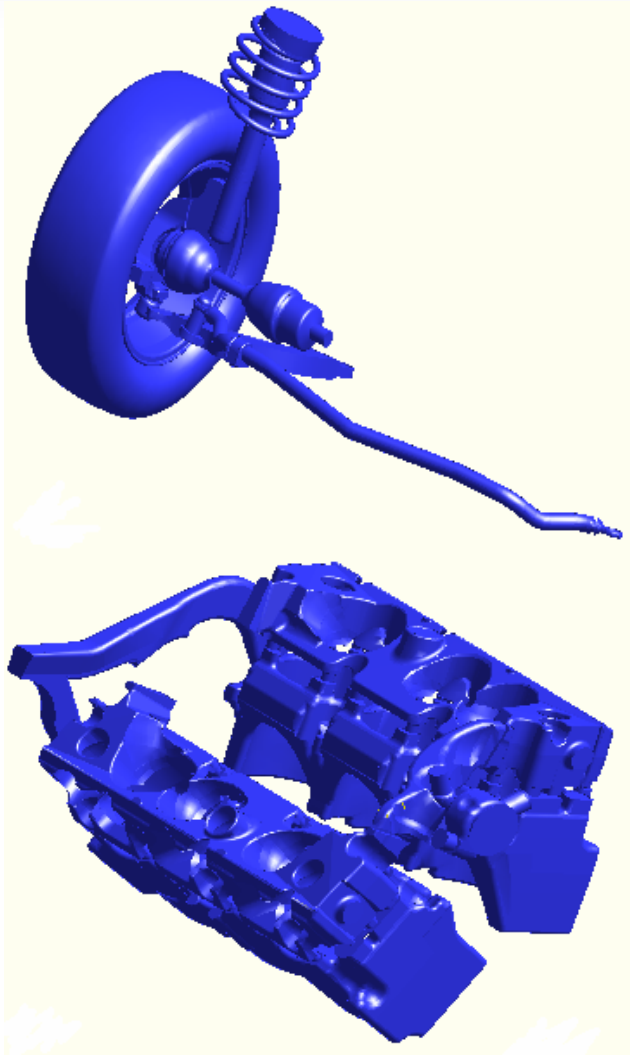
An Introduction to Geometry Design Algorithms

Fathi El-Yafi
Senior Product Engineer
Product Department of EXA Corporation

Geometry: Overview

- **Geometry Basics**
  - Definitions
  - Data
  - Semantic
- **Topology**
  - Mathematics
  - Hierarchy
- **CSG Approach**
- **BREP Approach**
  - Curves
  - Surfaces



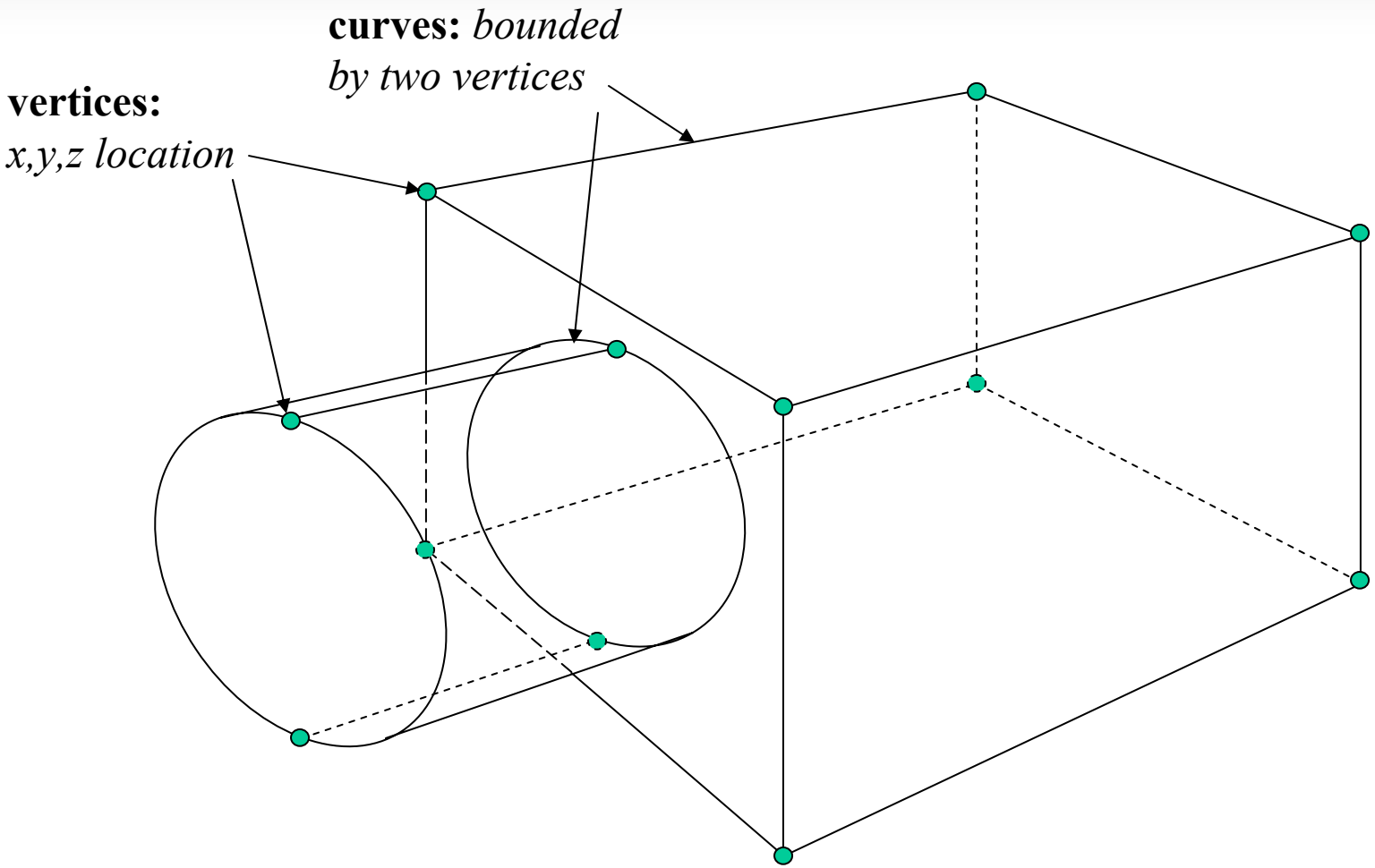
Geometry: Concept Basics

vertices:

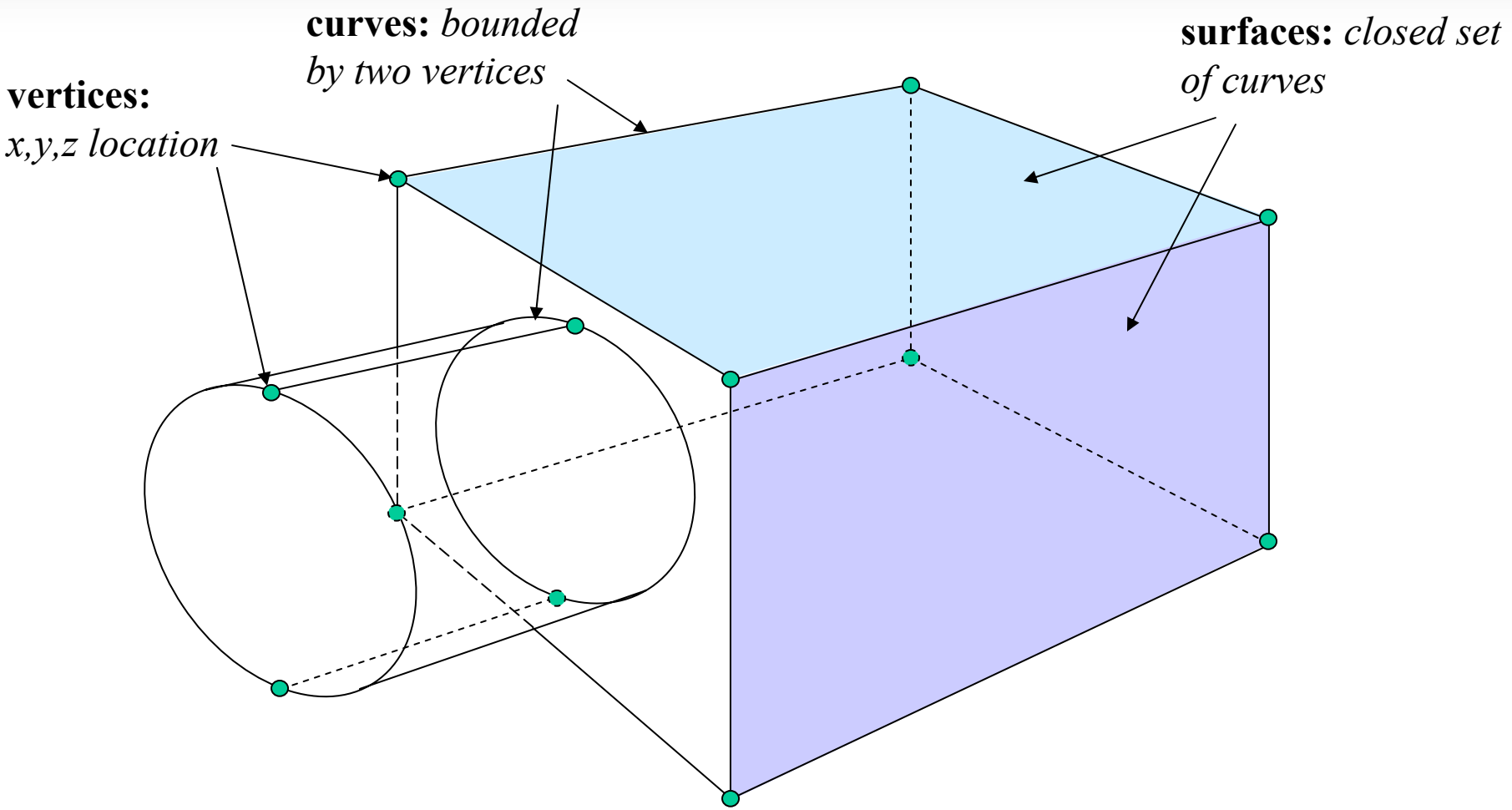
x,y,z location



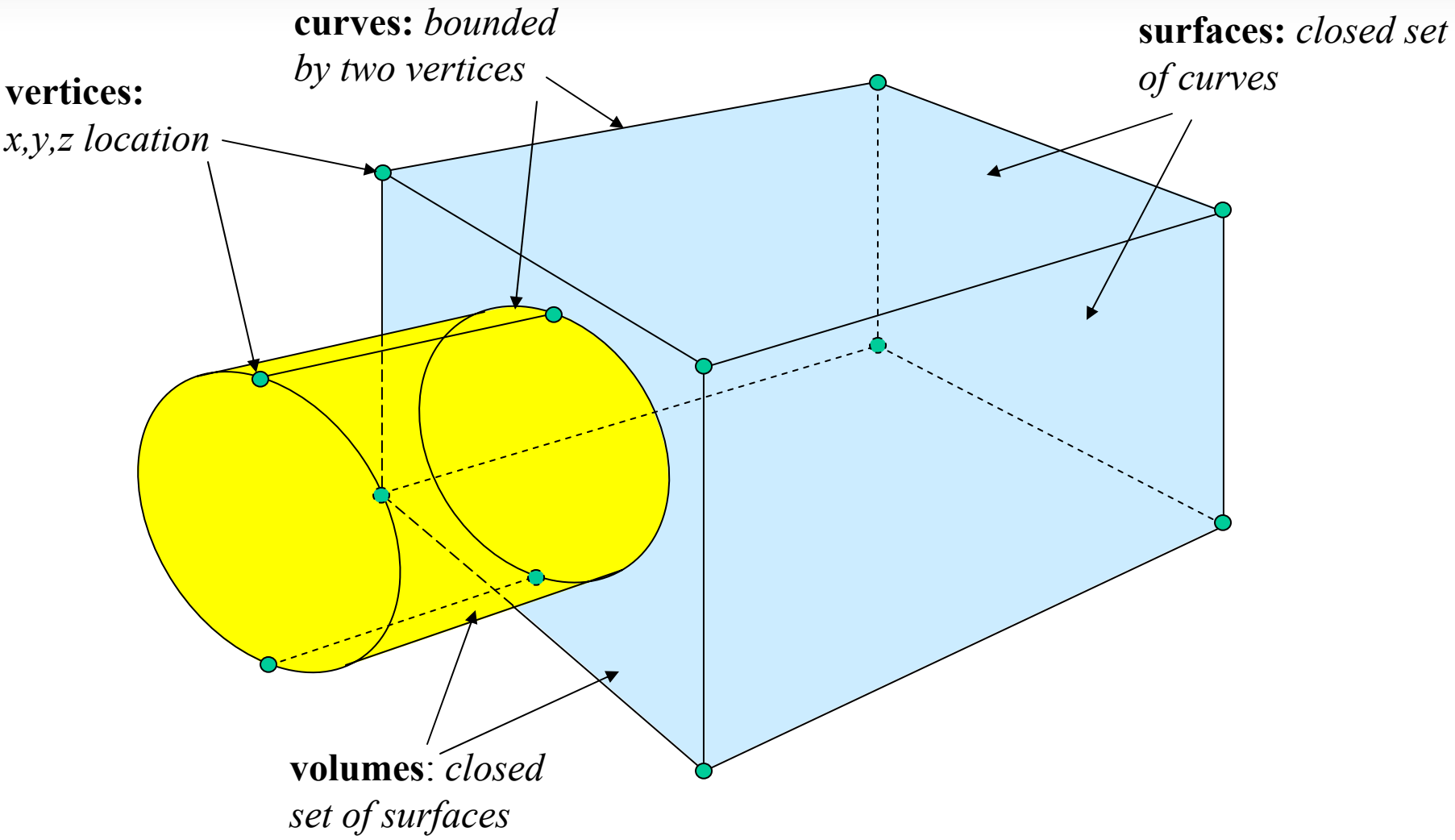
Geometry: Concept Basics



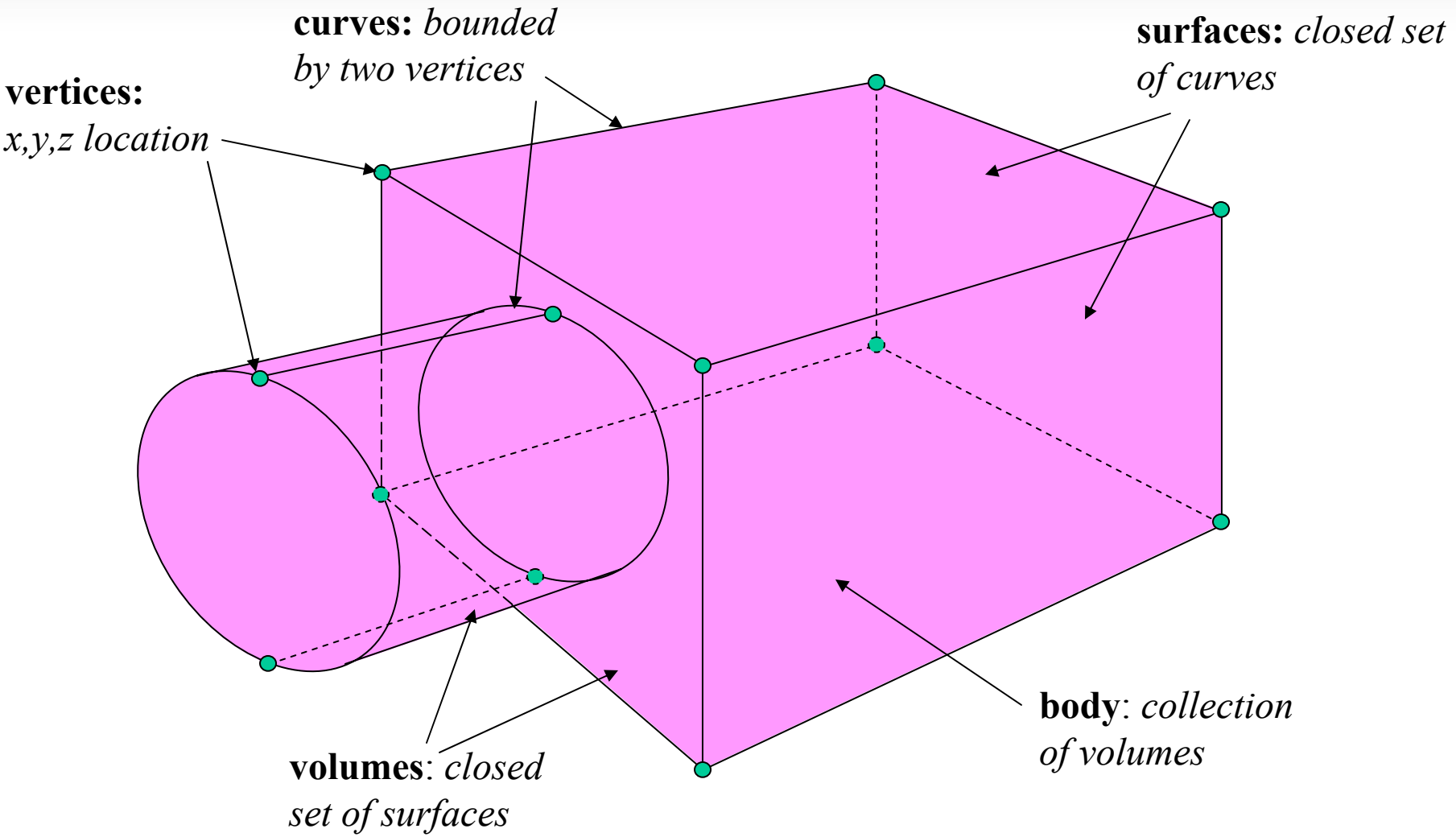
Geometry: Concept Basics



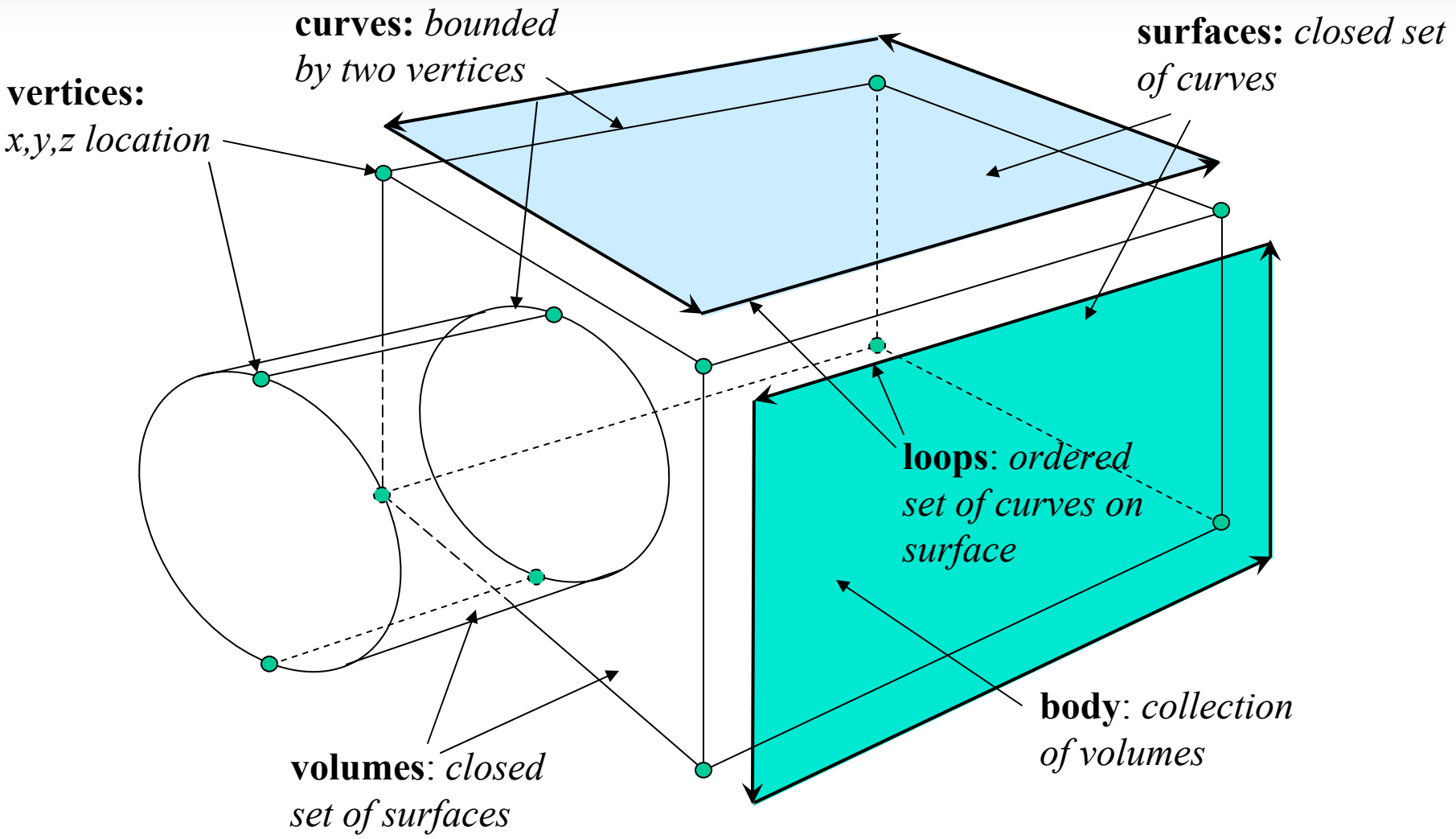
Geometry: Concept Basics



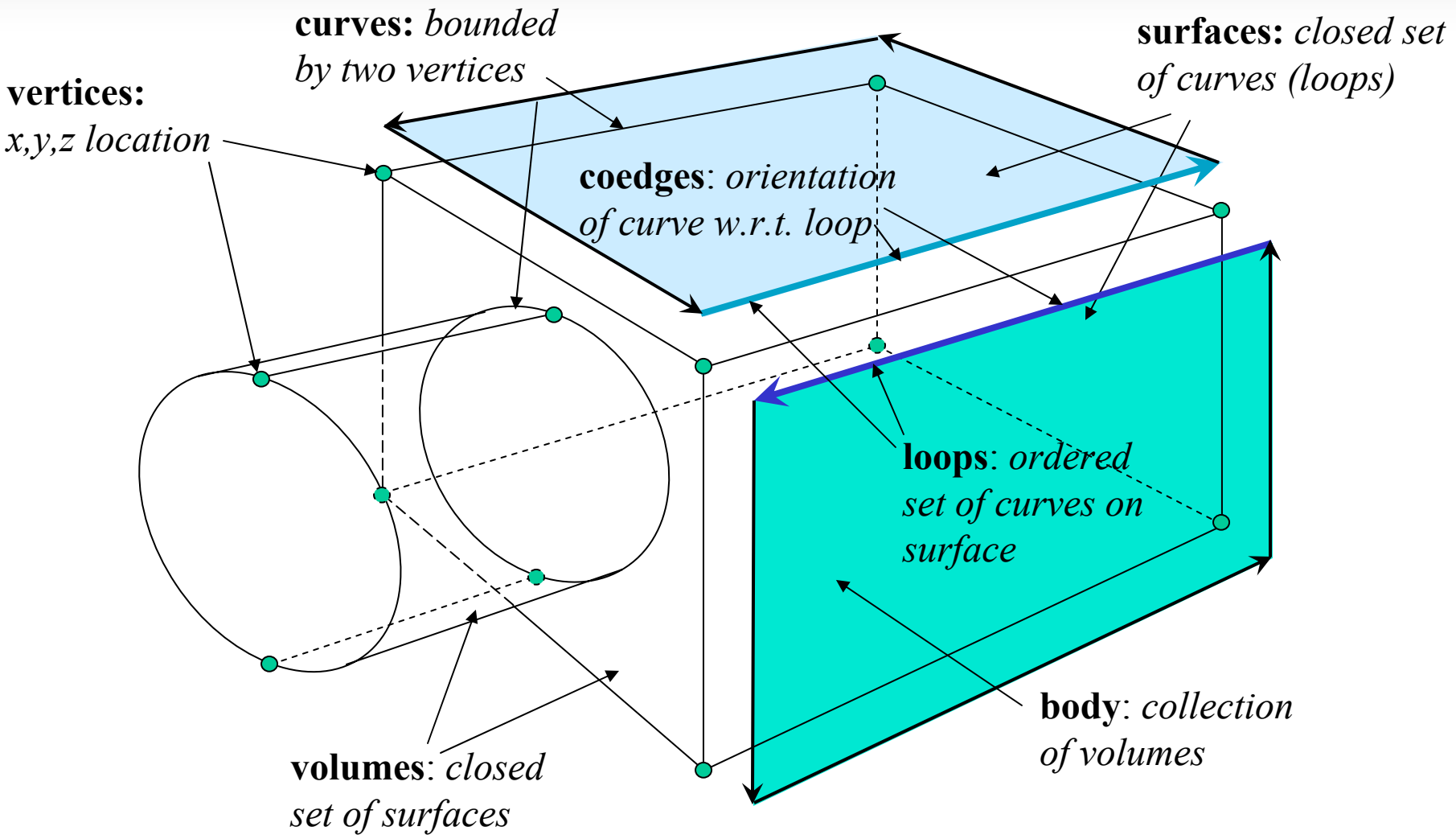
Geometry: Concept Basics



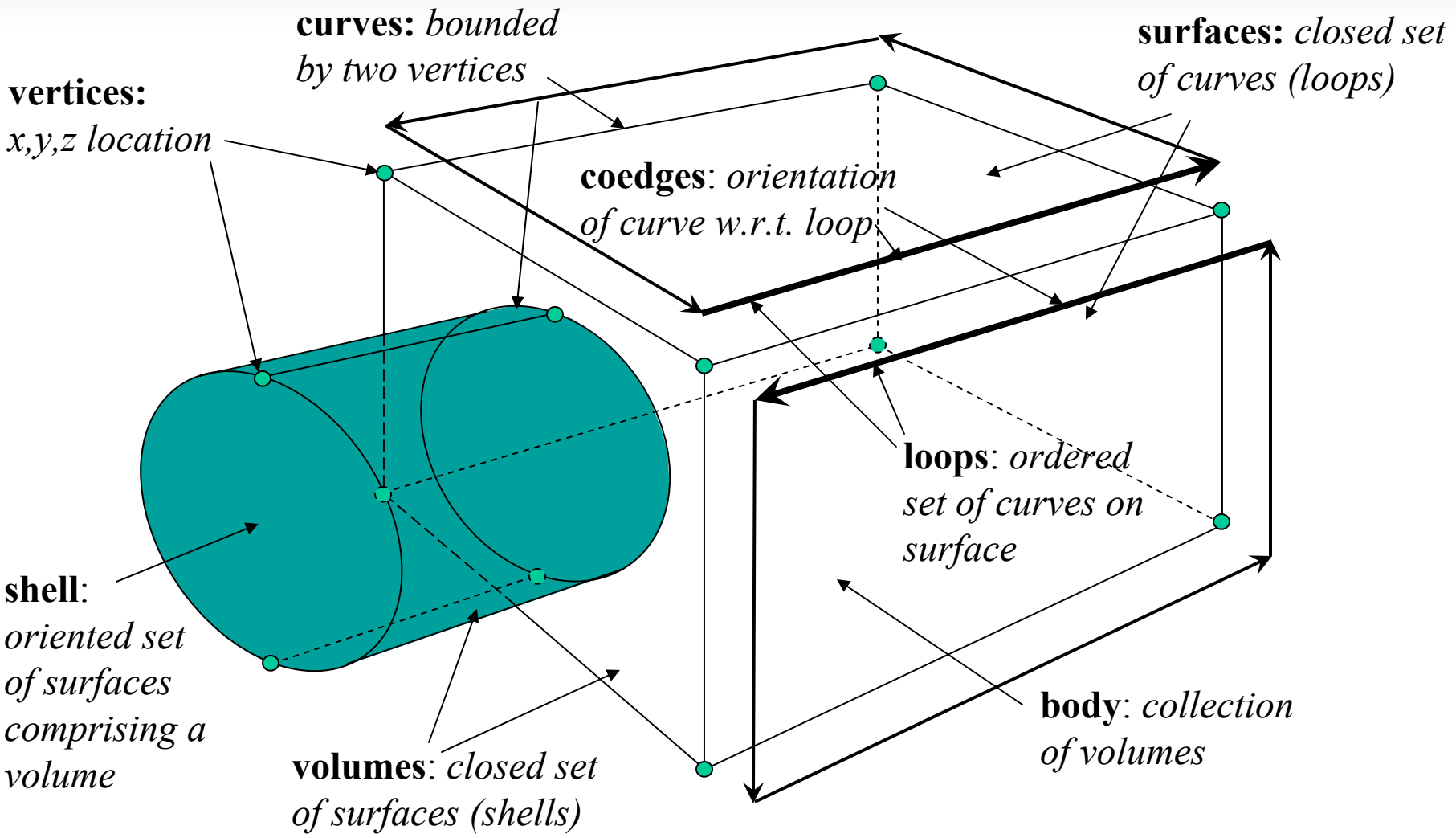
Geometry: Concept Basics



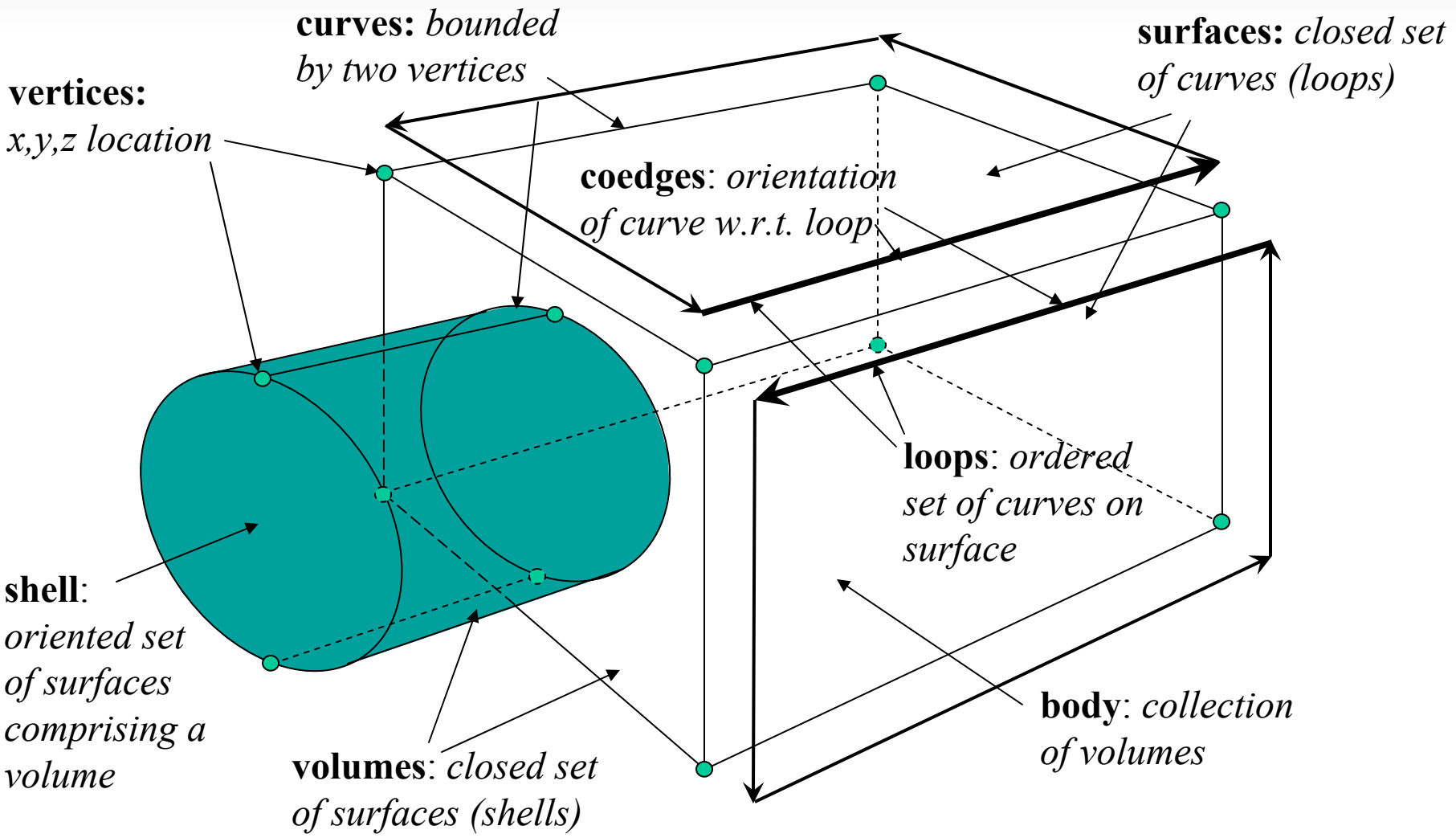
Geometry: Concept Basics



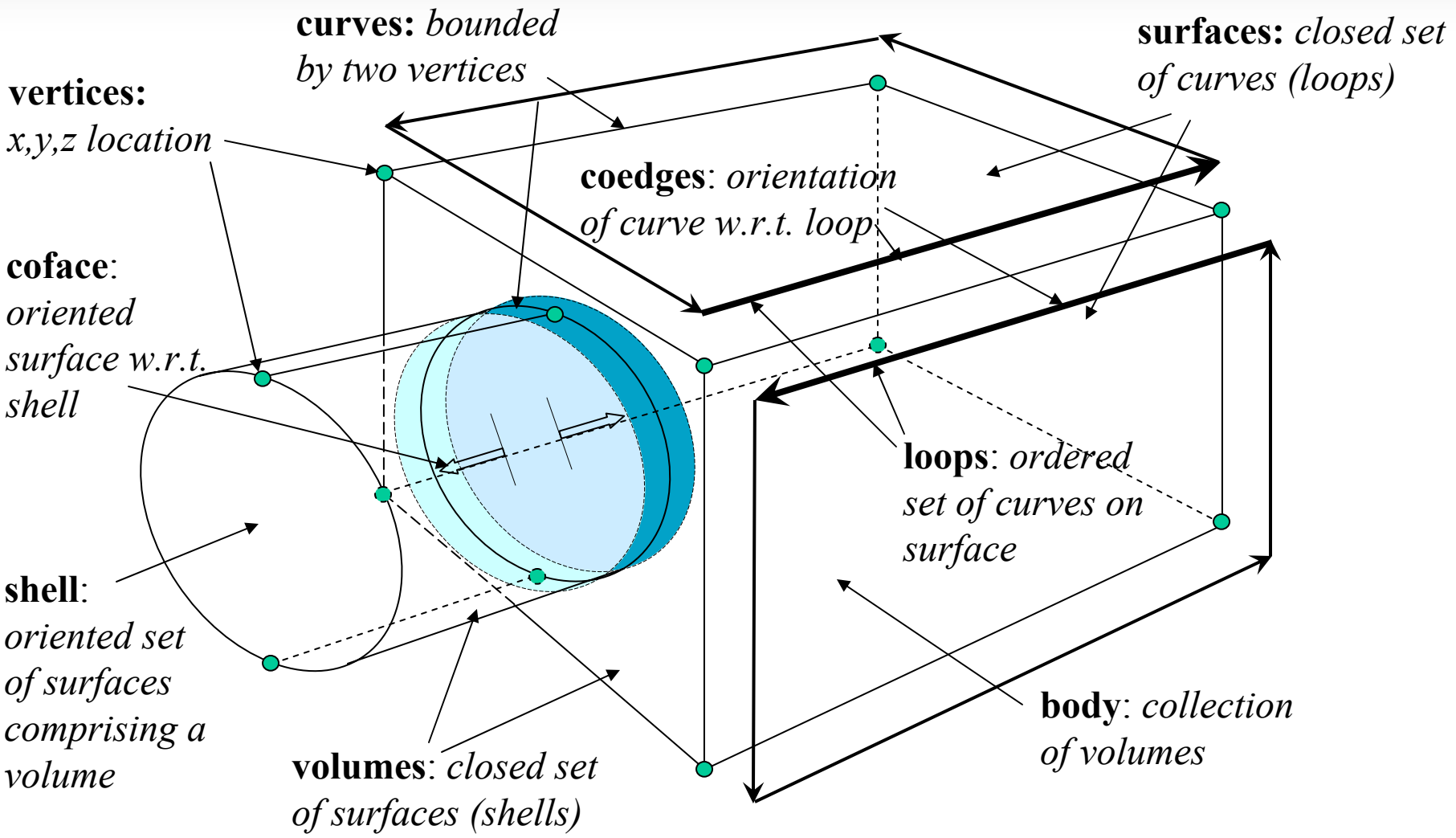
Geometry: Concept Basics



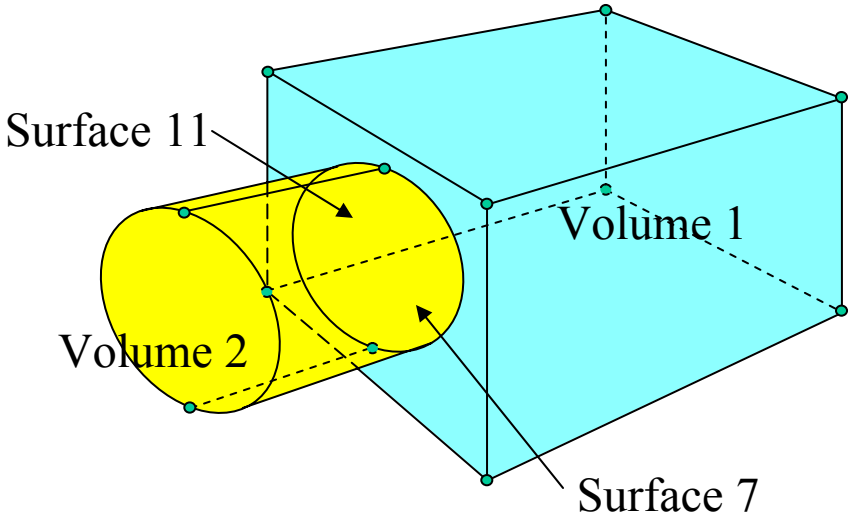
Geometry: Concept Basics



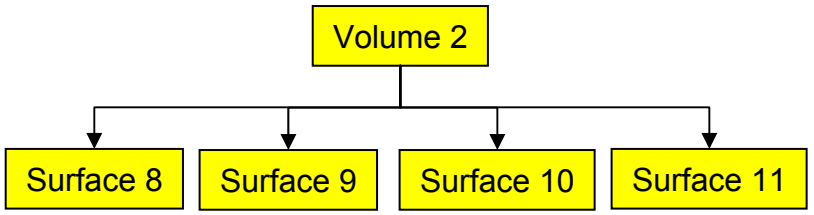
Geometry: Concept Basics

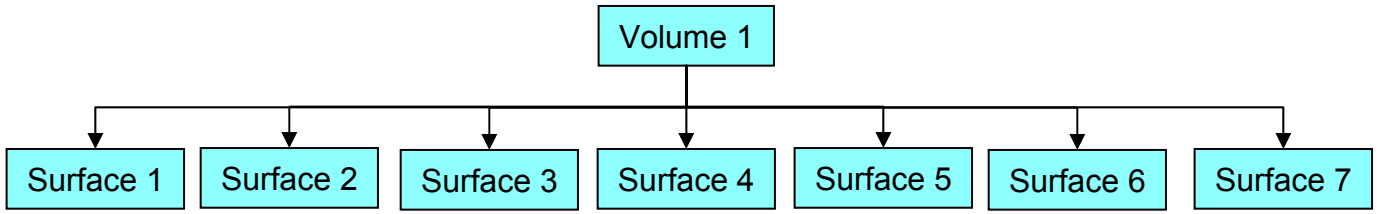


Geometry: Concept Basics

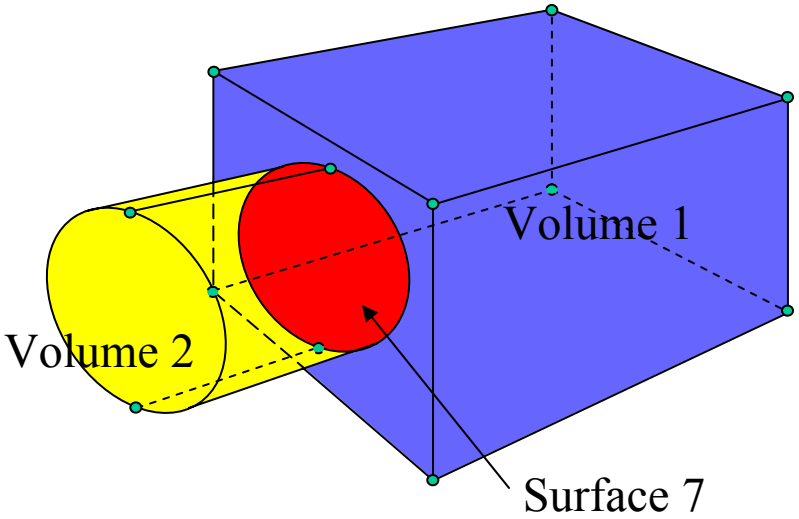


Manifold Geometry:
Each volume maintains its own set of unique surfaces

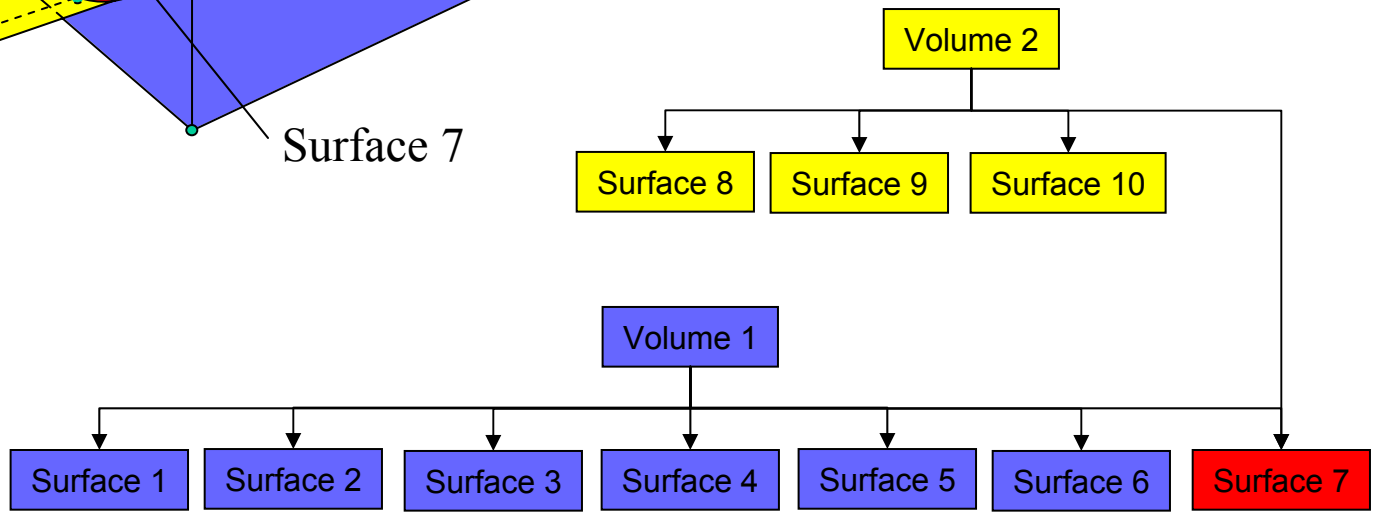




Geometry: Concept Basics



Non-Manifold Geometry:
Volumes share matching surfaces

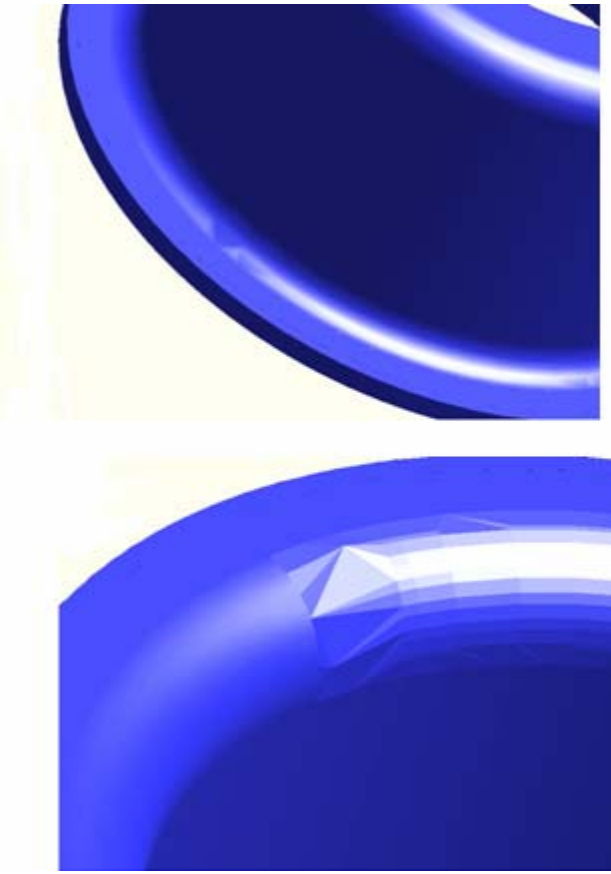


Geometry: Data

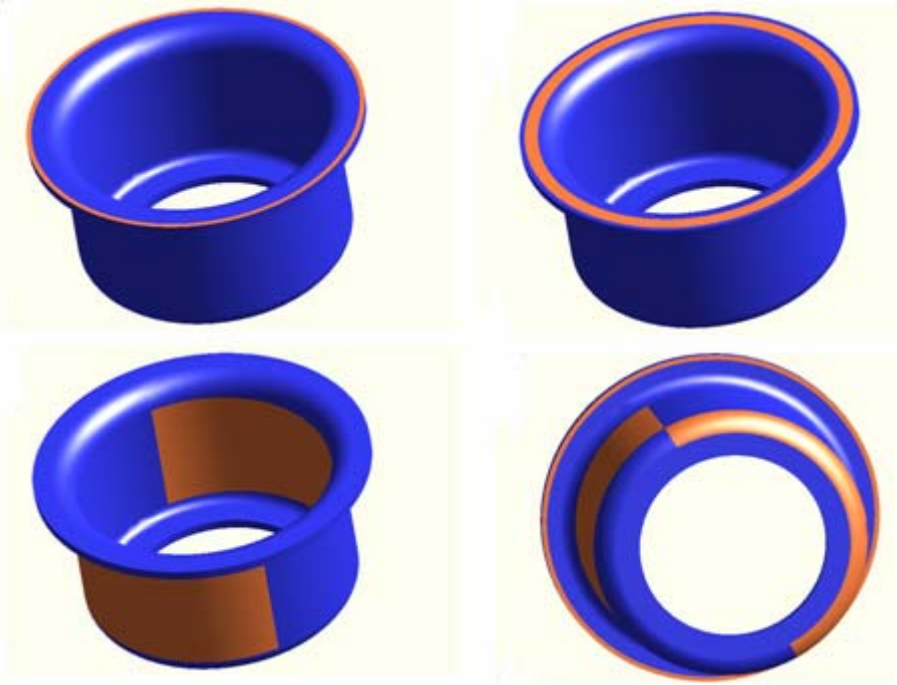
- **Model**
  - Wireframe
  - Surface
  - Volume
- **Geometrical Data Separated from Attributes**
- **Attributes**
  - Colors
  - Parameters
  - Etc.
- **Graphical Objects**
- **Visible Parts**

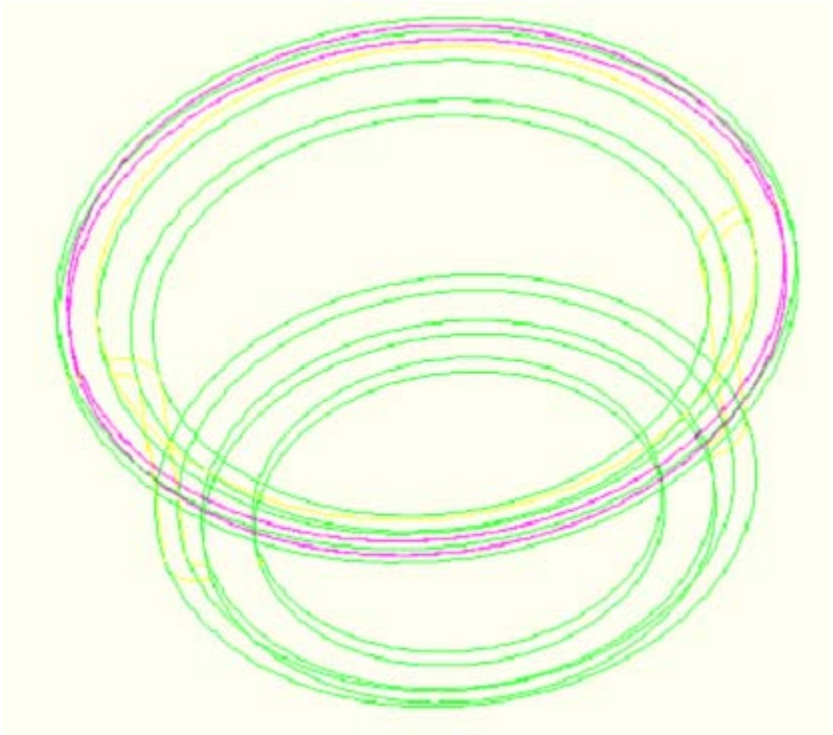
Semantic: Surface-Volume-Deflection-Defects



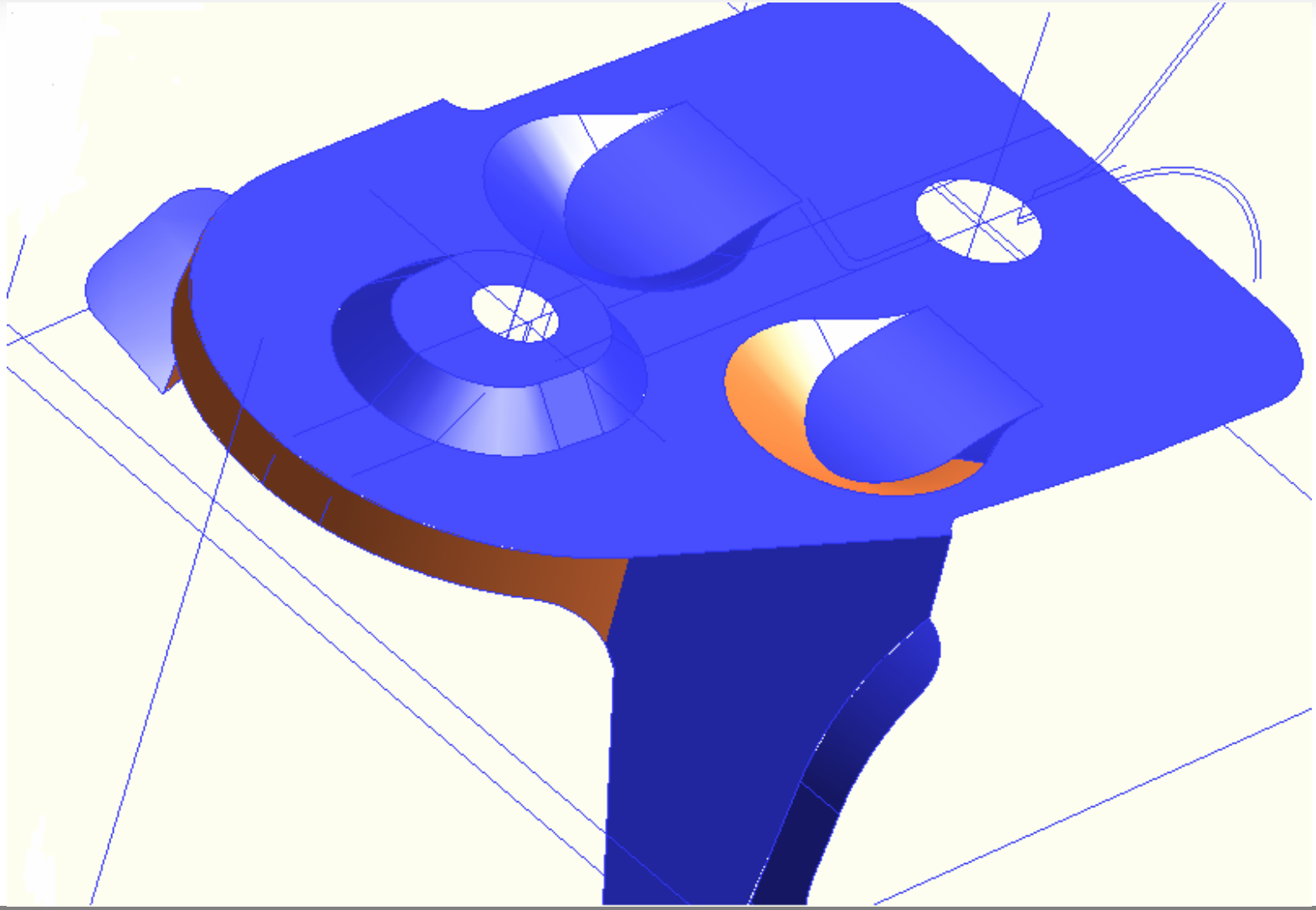


Semantic: Surfaces and Features

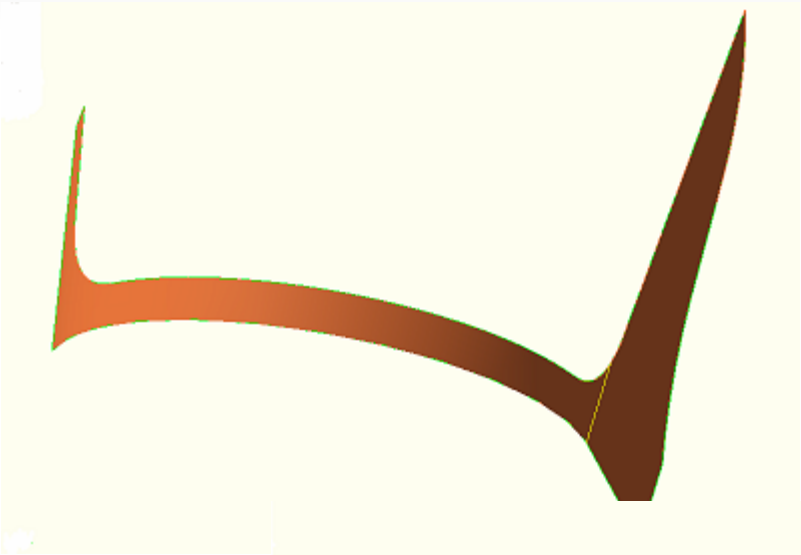


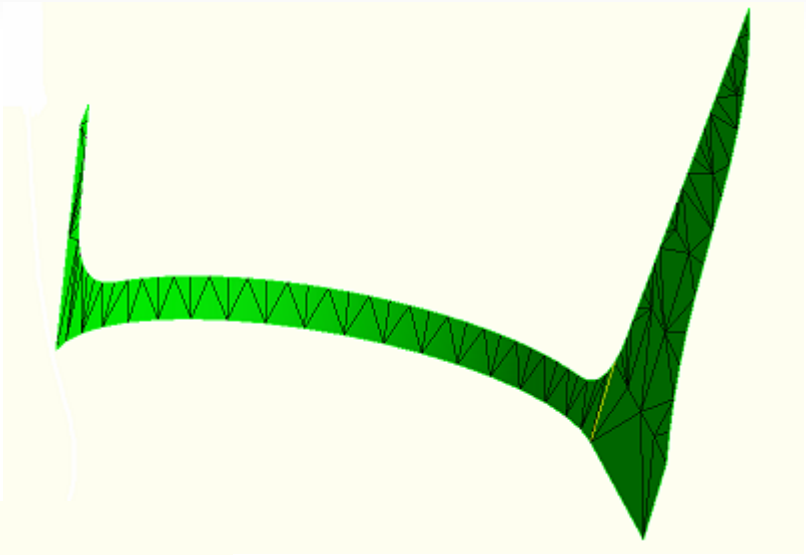


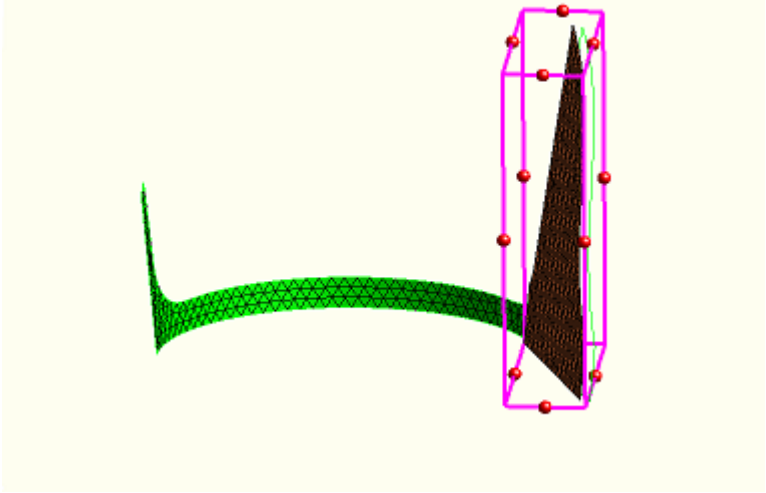
Semantic: Detail



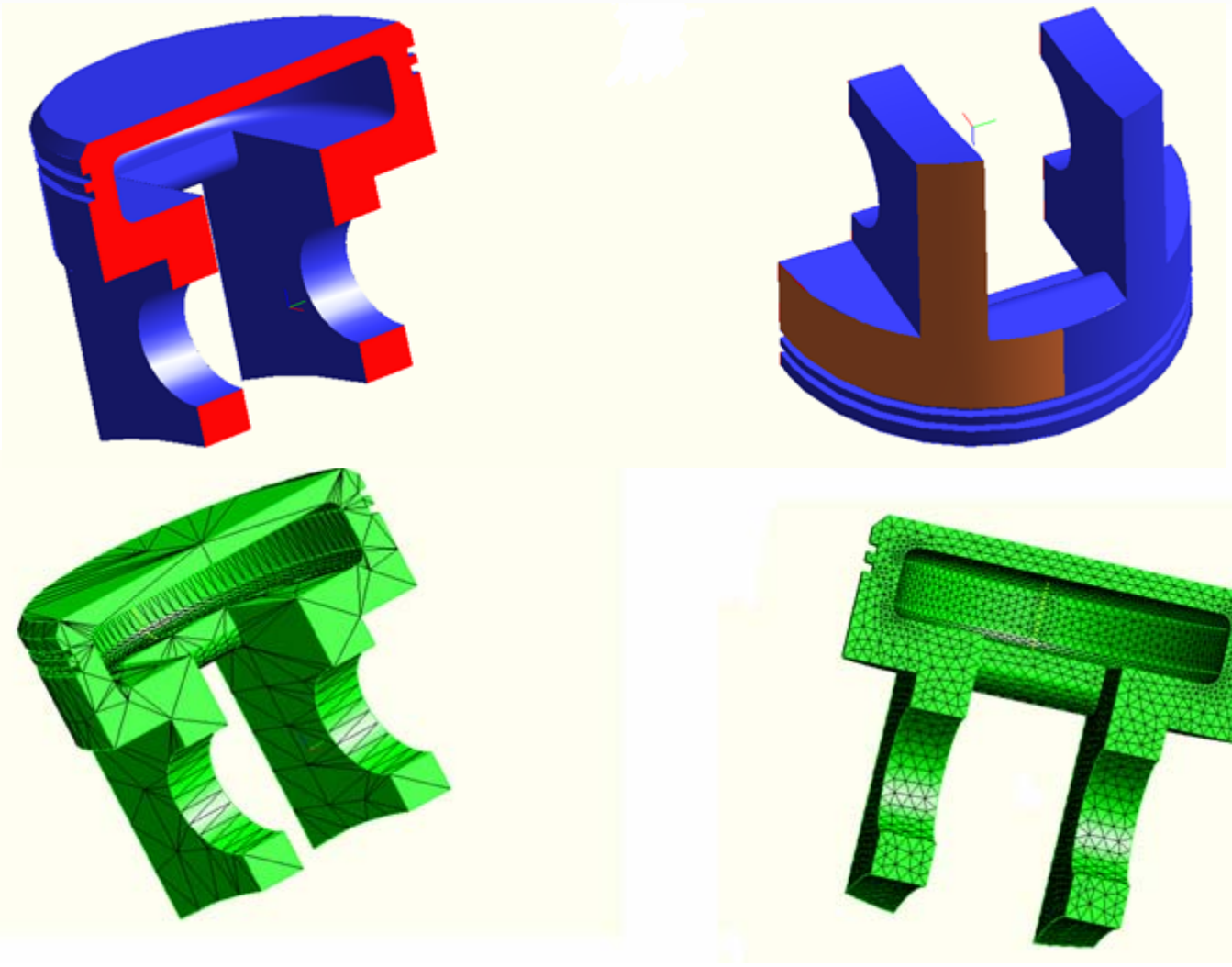
Semantic: Detail-Mesh–Bounding Box



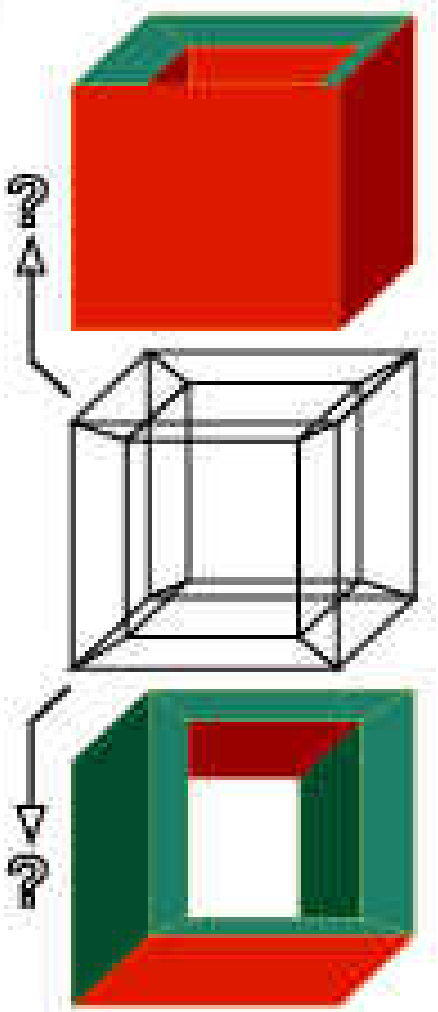




Semantic: Decomposition–Curvature Mesh STL-Mesh FEM



Wireframe Model: Limits

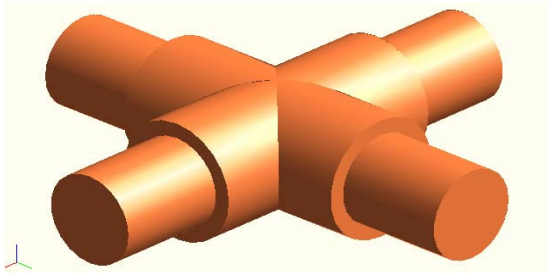


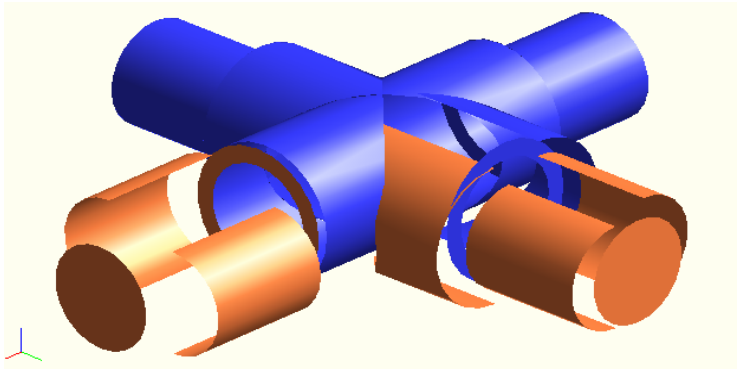
Topology?

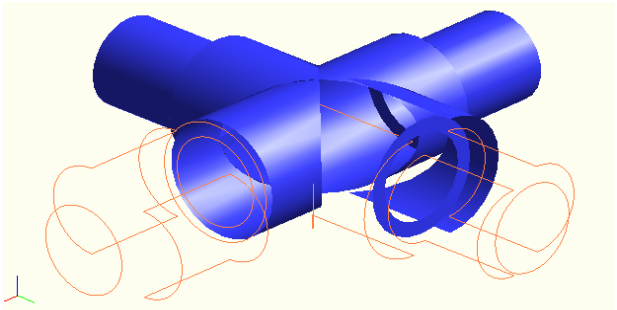
- Concept of interior and exterior
- Orientation
- Reliable calculation of basic parameters:
  volume, center of gravity, axis of inertia …
- Contour = oriented surface and limited area
- Tree construction

Relations: (Geometry, Topology)

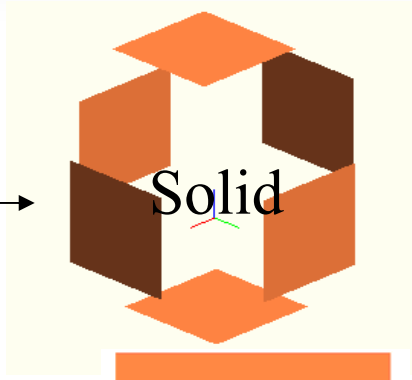
Geometry

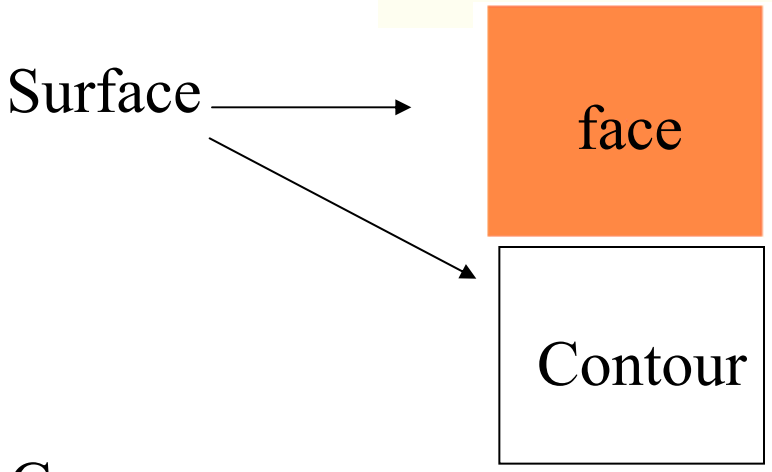


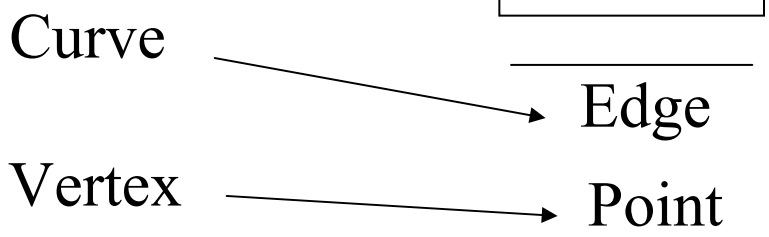




Topology



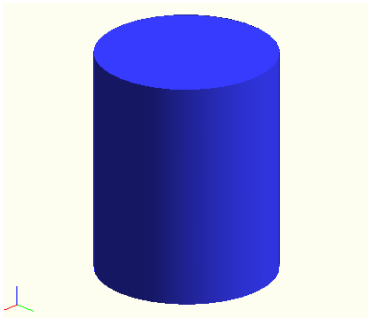




Topology: Mathematics

A contour “grips" to its interior

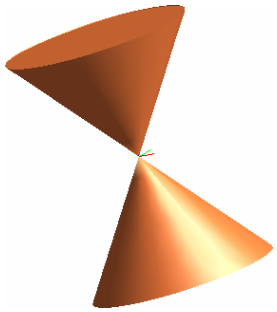
Either S is a **Set** of R^3



Adhesion : A(S)

$P \in A(S)$: any neighborhood of P contains a point of S

Interior : I(S)



$P \in I(S)$ if $\exists\ V(P) \subset S$

Boundary : B(S)

$P \in B(S)$ if $P \in A(S)$ and $P \in A(C(S))$
where C(S) refers to the complementary of S in R^3

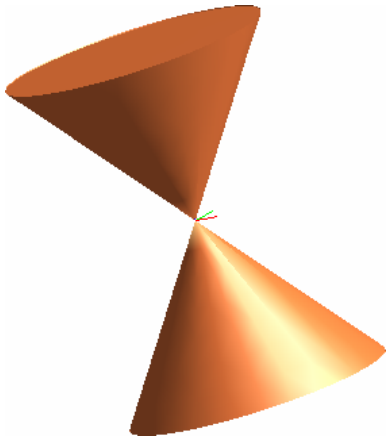
Topology: Mathematics

Open : S = A(S)

Closed : S = I(S)

Regular Solid

S = A(I(S)) : adhesion of its interior = R(S)



Topology: Mathematics

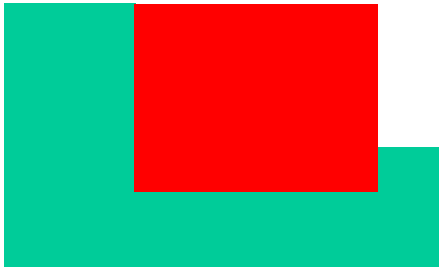
Operator of Regularization

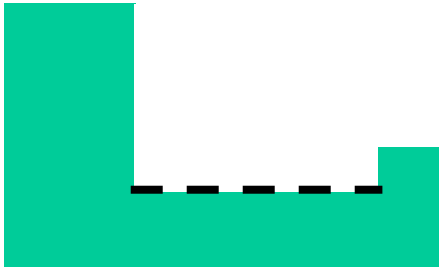
$A \cap B = R(A \cap B)$

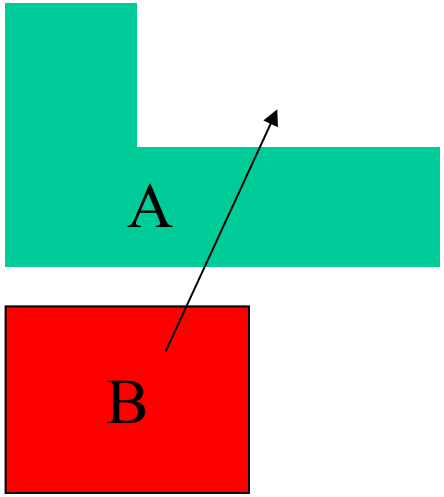
$A \cup B = R(A \cup B)$

$A - B = R(A-B)$

$C(A) = R(C(A))$









A - B

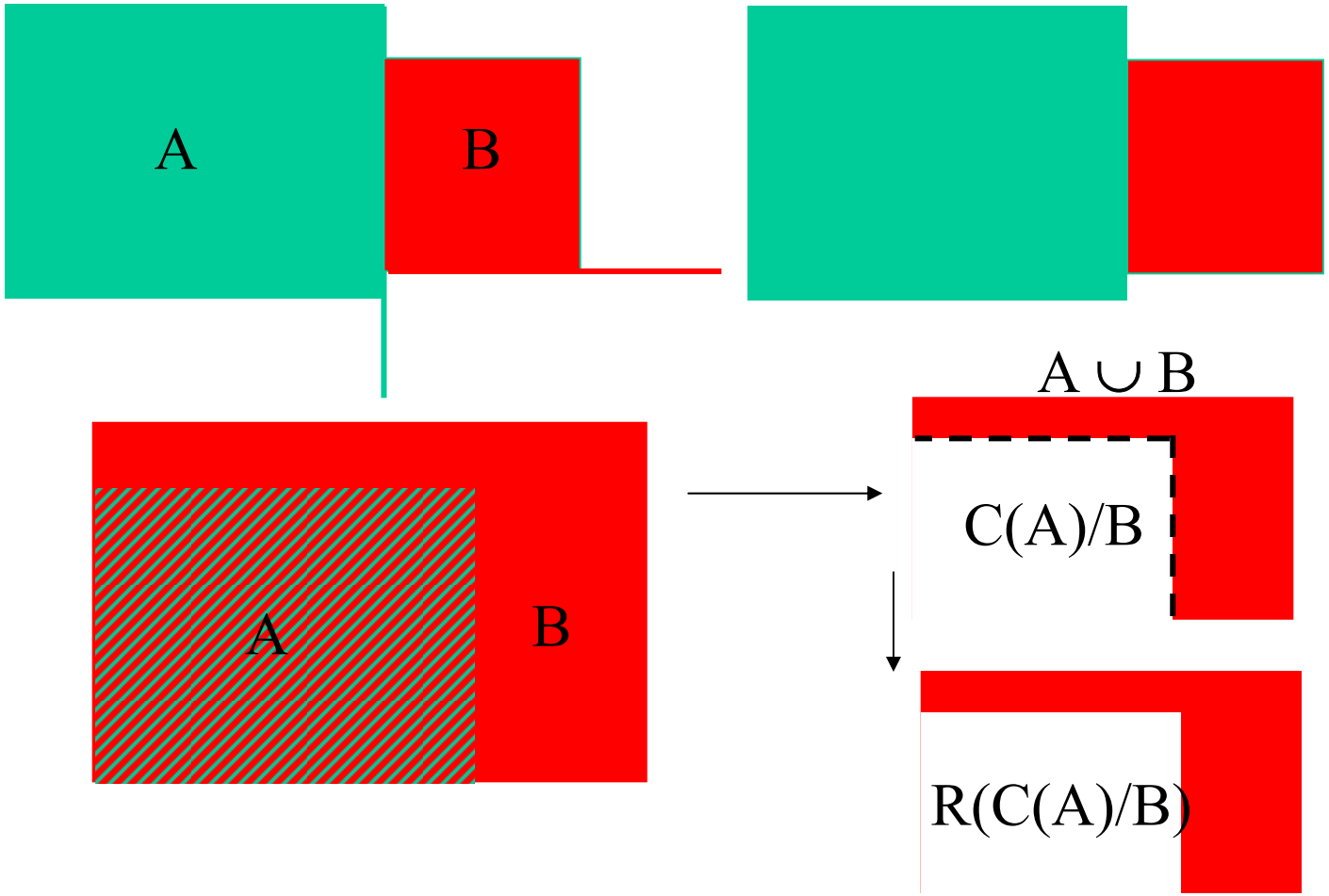
R(A ∩ B)





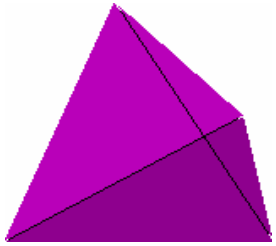
R(A – B)

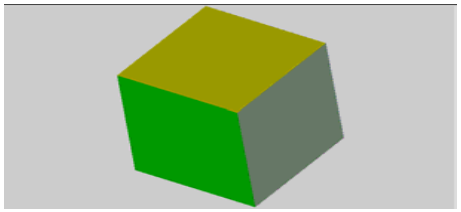
Operator of Regularization



Topology: Mathematics

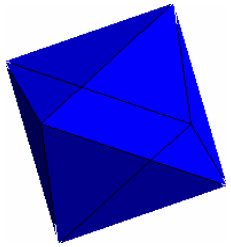
Euler Formula: V+F = E+2

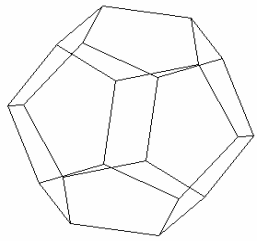


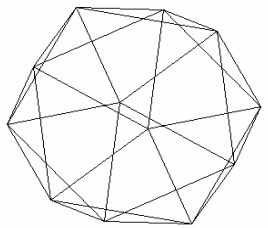


Polyhedron	Type of Faces	F	V	E
Tetrahedron	Equilateral Triangles	4	4	6
Octahedron	Equilateral Triangles	8	6	12
Cube	Squares	6	8	12
Dodecahedron	Pentagons	12	20	30
Icosahedrons	Equilateral Triangles	20	12	30

F = number of faces,
V = number of vertices
E = number of edges

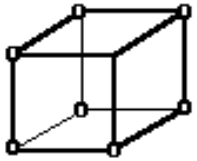
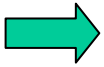
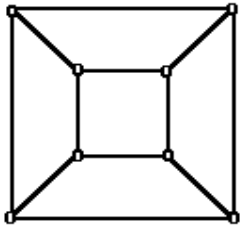
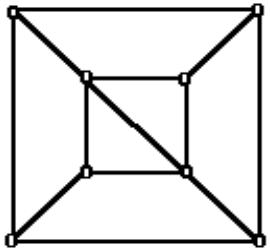
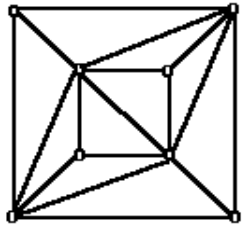




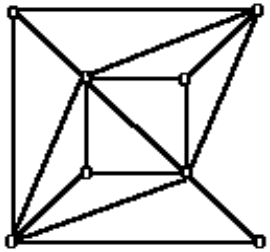
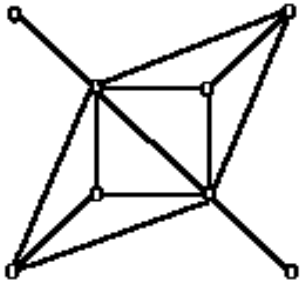
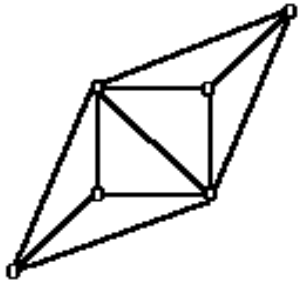
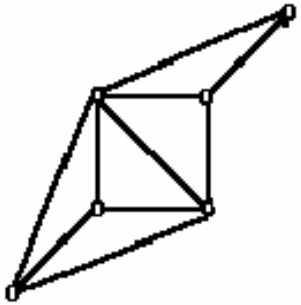


Topology: Mathematics

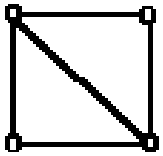
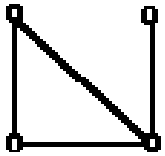
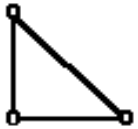
Cauchy Proof (1789-1857)

V+F=E+2 V+F=E+1 V+F+1=(E+1)+1 V+F=E+1

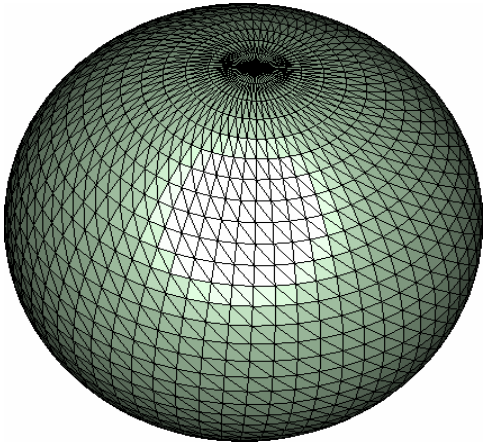
V+F=E+1 V+F=E+1 (V-2)+F=(E-2)+1 V+F=E+1

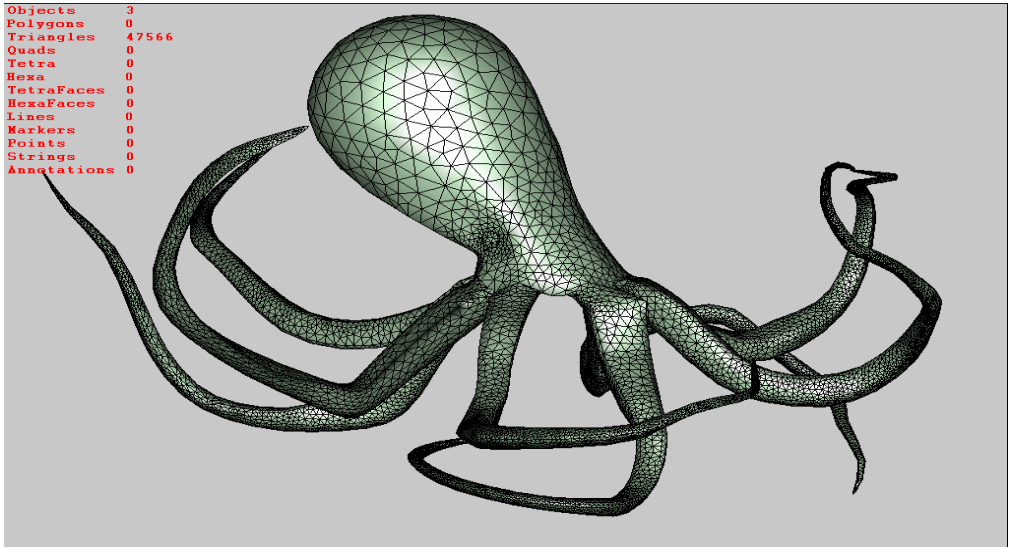
3+1=3+1!!

Topology: Mathematics

Mesh Examples

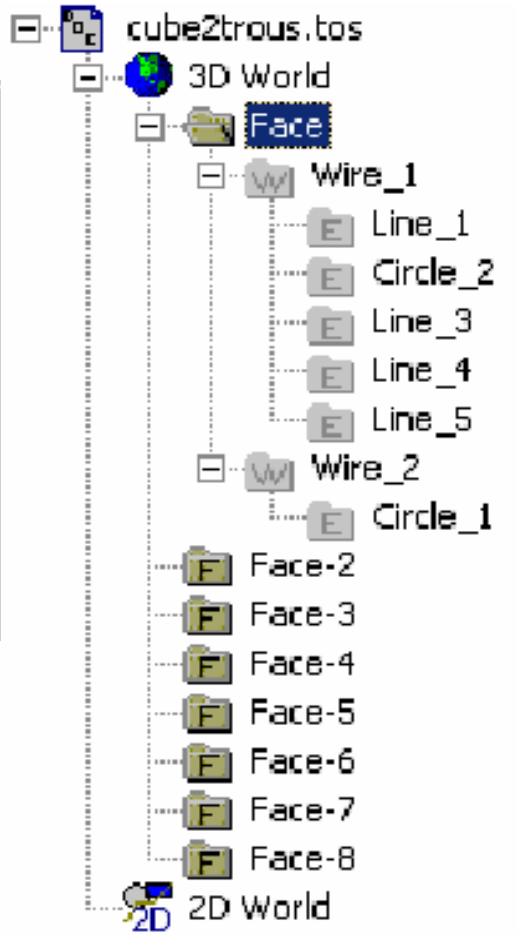


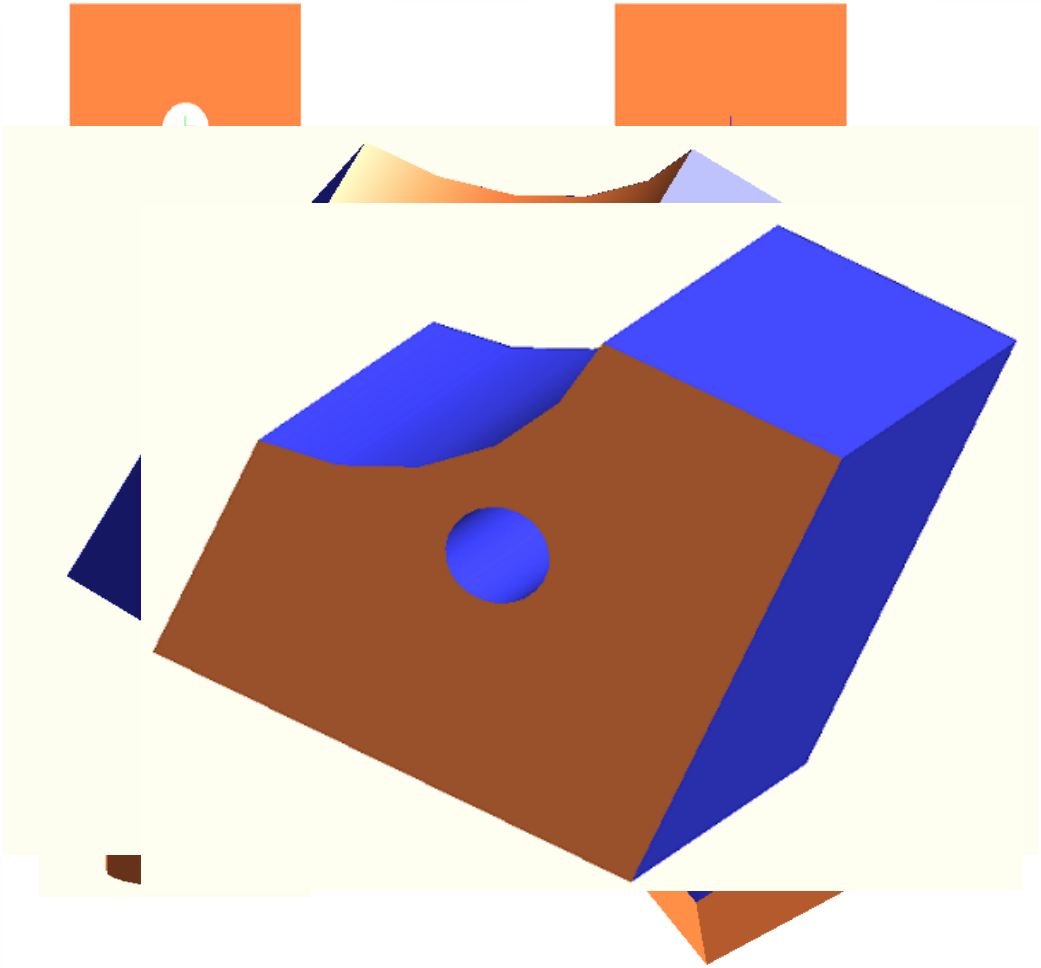
F = 3844
V = 1924
E = V + F –2 = 5766



F = 47566
V = 23793
E = S + F –2 = 71357

Topology : Hierarchy





CSG : Constructive Solid Geometry

Constructive Representation

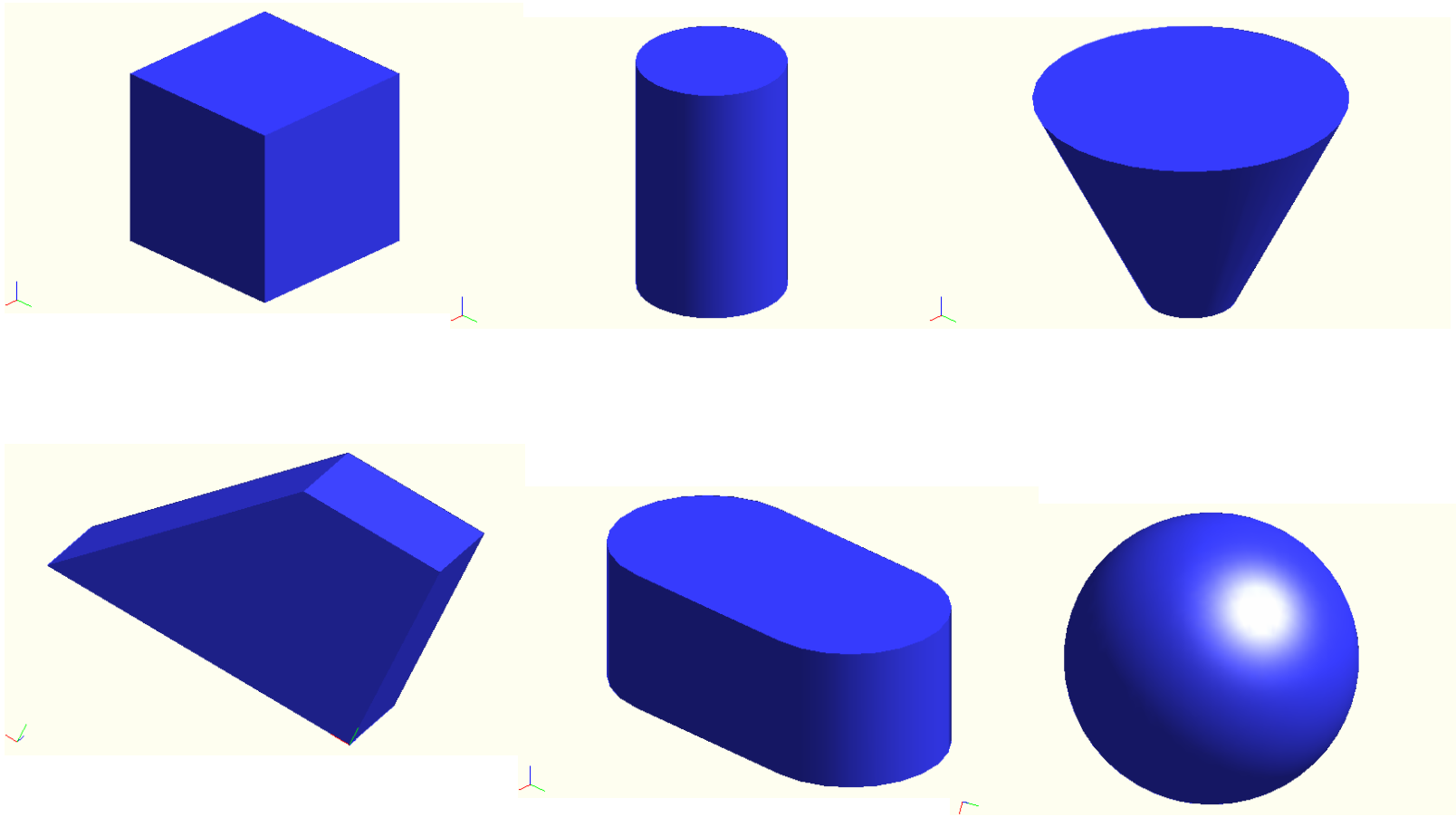
- Parametric Volume Primitives
- Transformations
- Boolean Operators:
  - Union, common, subtract

Concept of Graph

Advantage :

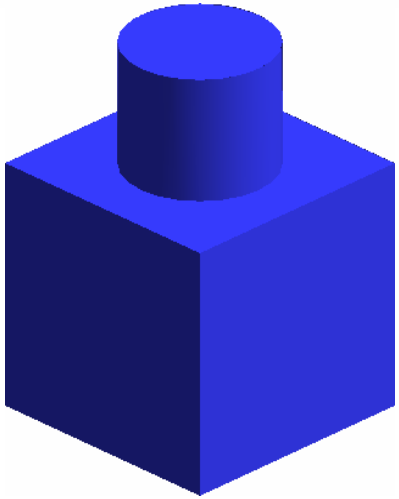
- Simple Description
- Simulation of Object « Manufacturing »

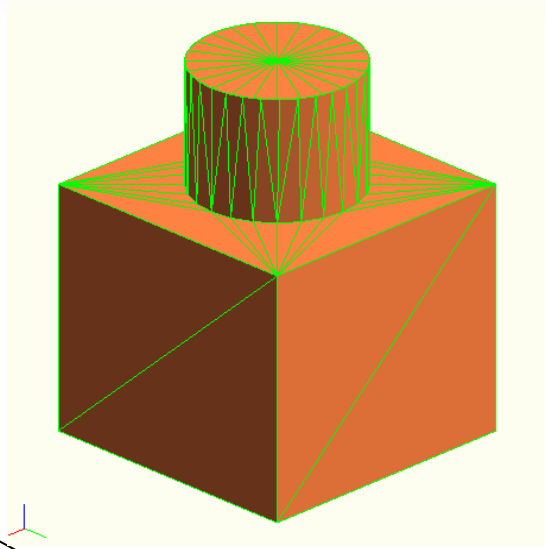
CSG: Primitive Components



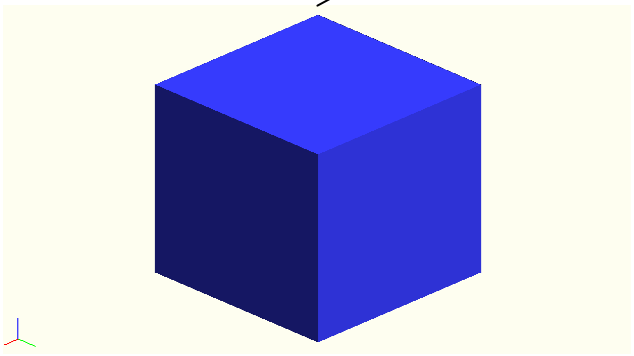
CSG: Boolean Operators

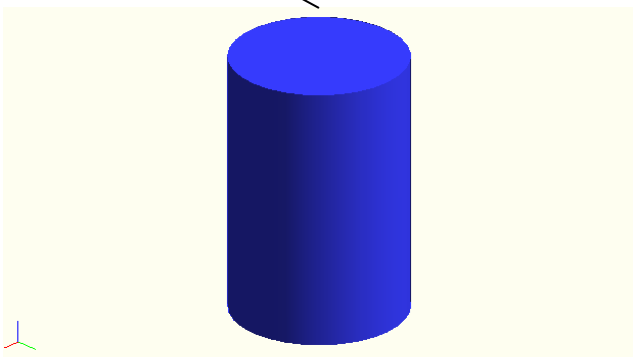
Union





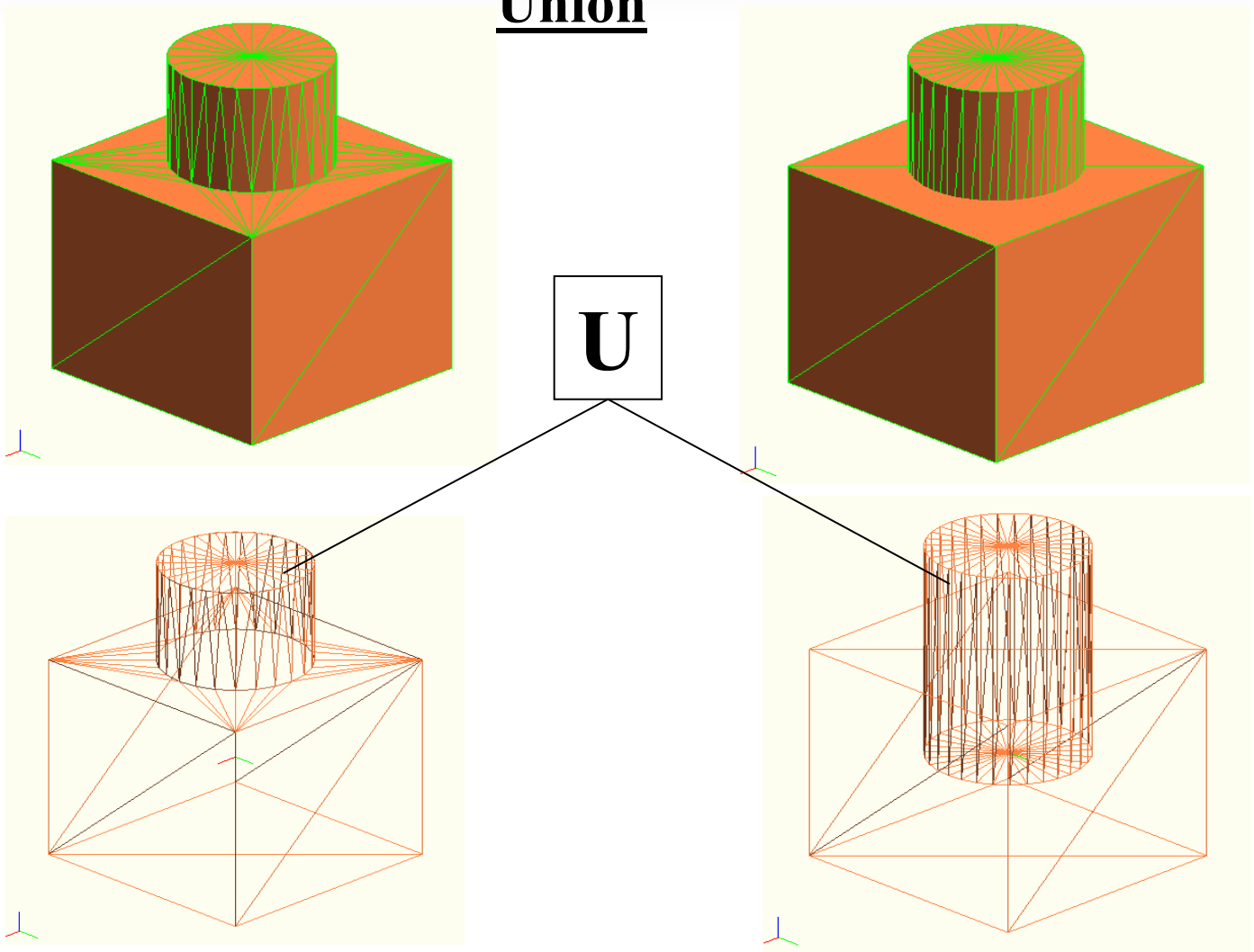
U





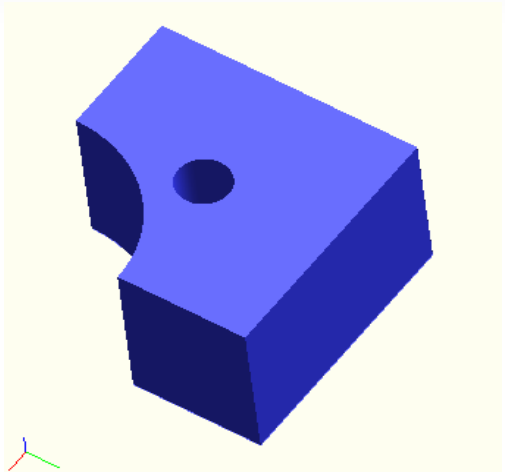
CSG: Boolean Operators

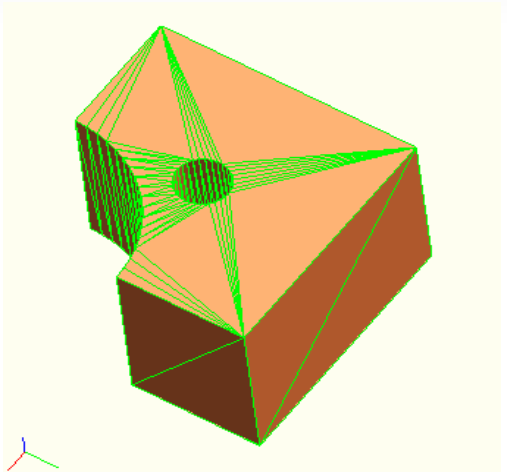
Union

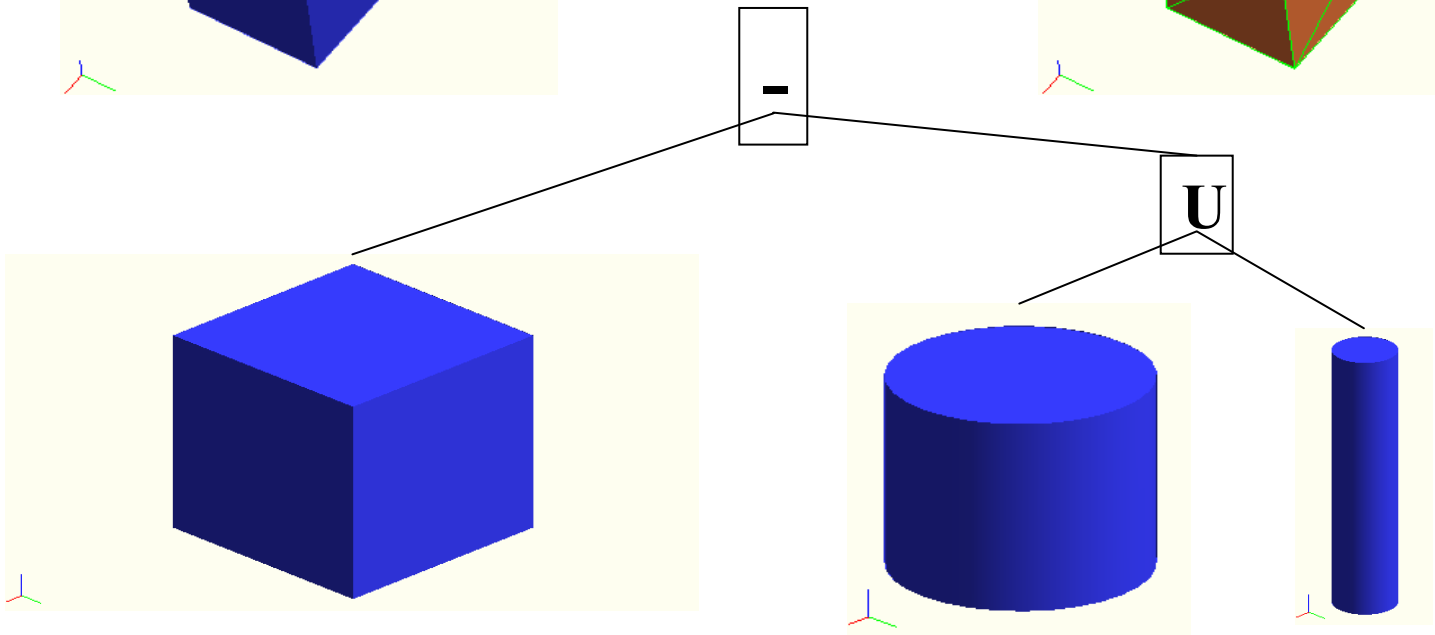


CSG: Boolean Operators

Subtract

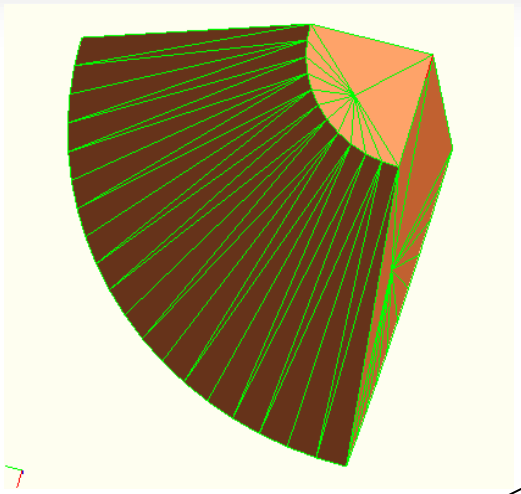


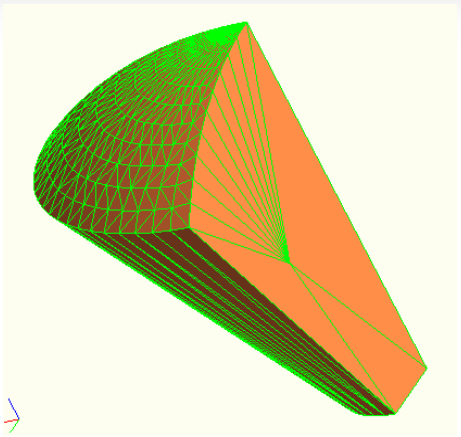


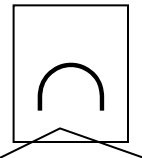


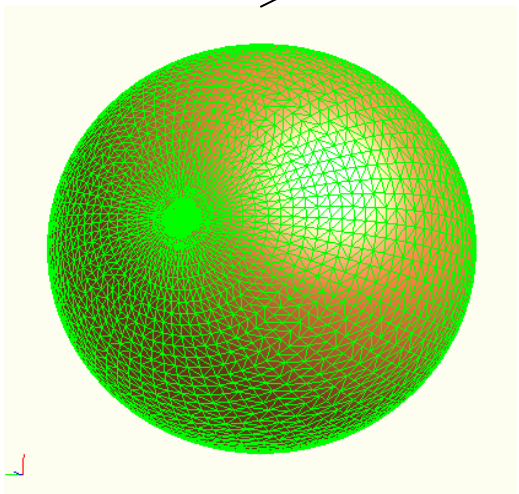
CSG: Boolean Operators

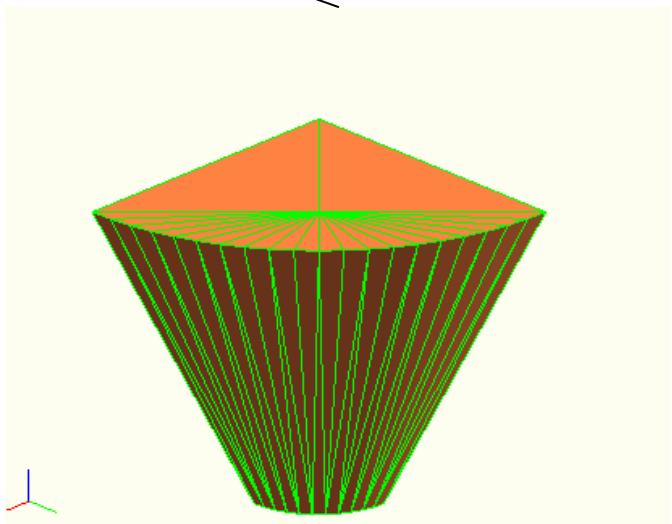
Common





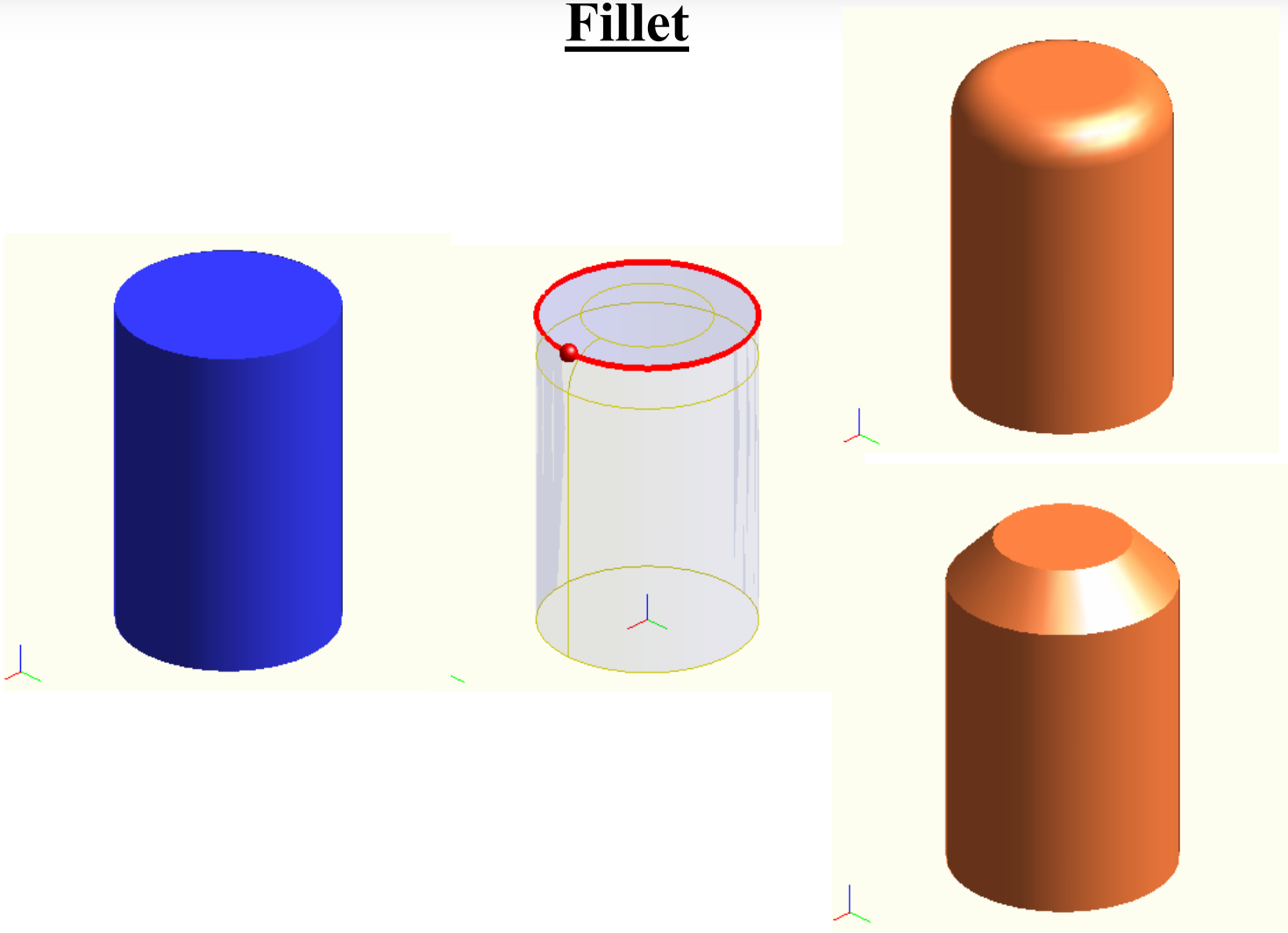






CSG: Boolean Operators

Fillet

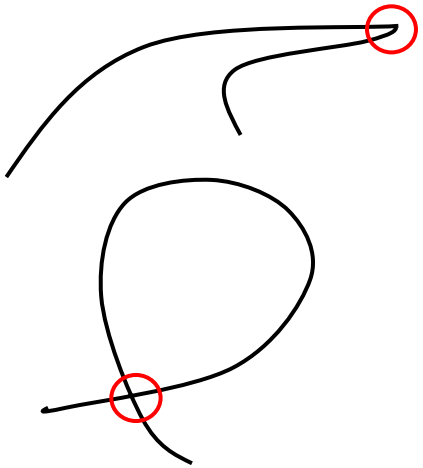


Geometry: Curves

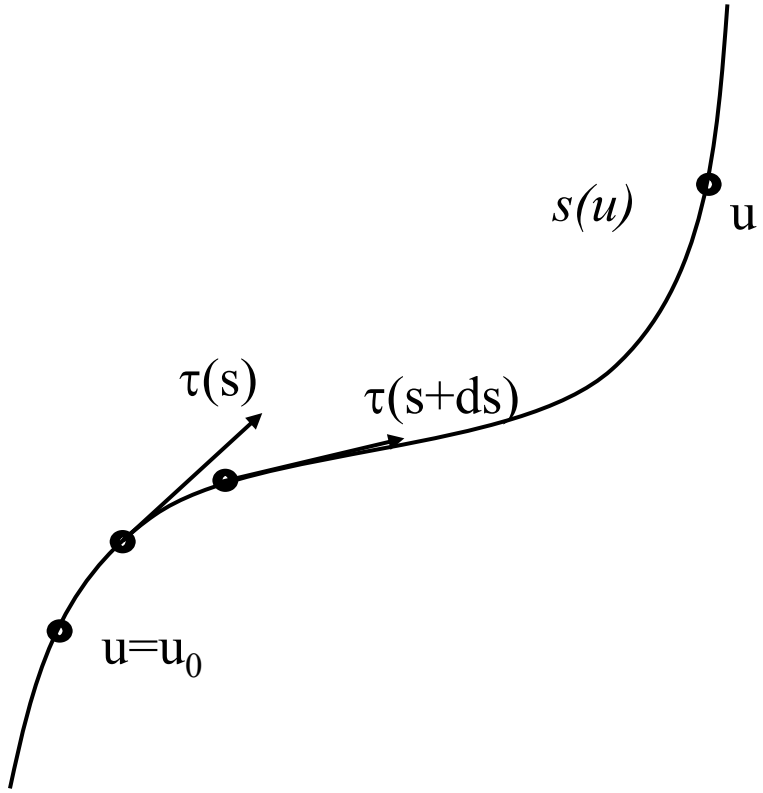
$$\overrightarrow{OM(u)} \begin{cases} x(u) \\ y(u) \\ z(u) \end{cases}$$

x, y, z function of u, continuous first order

Lap back point



Multiple point

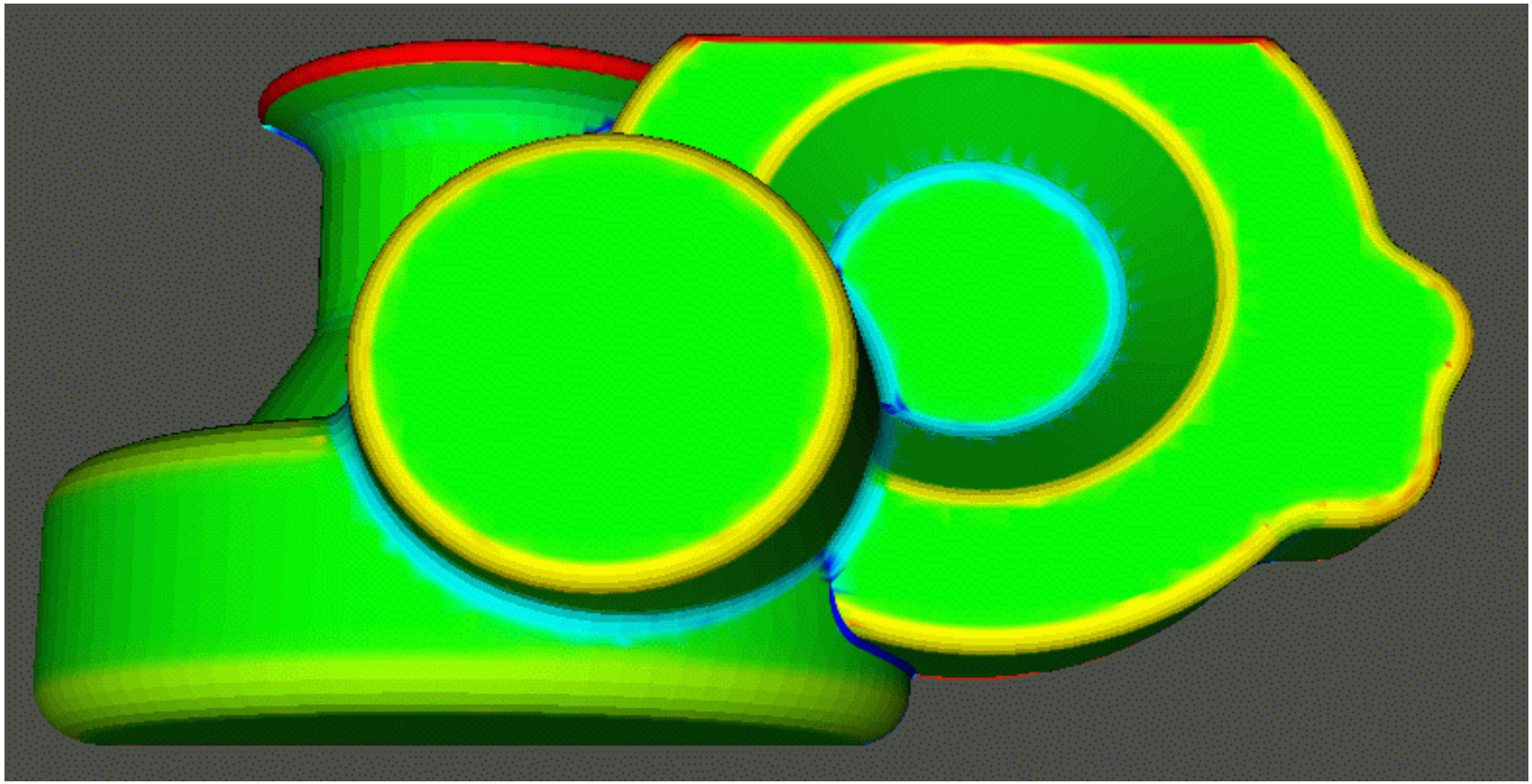


Arc length, abscissa curvilinear

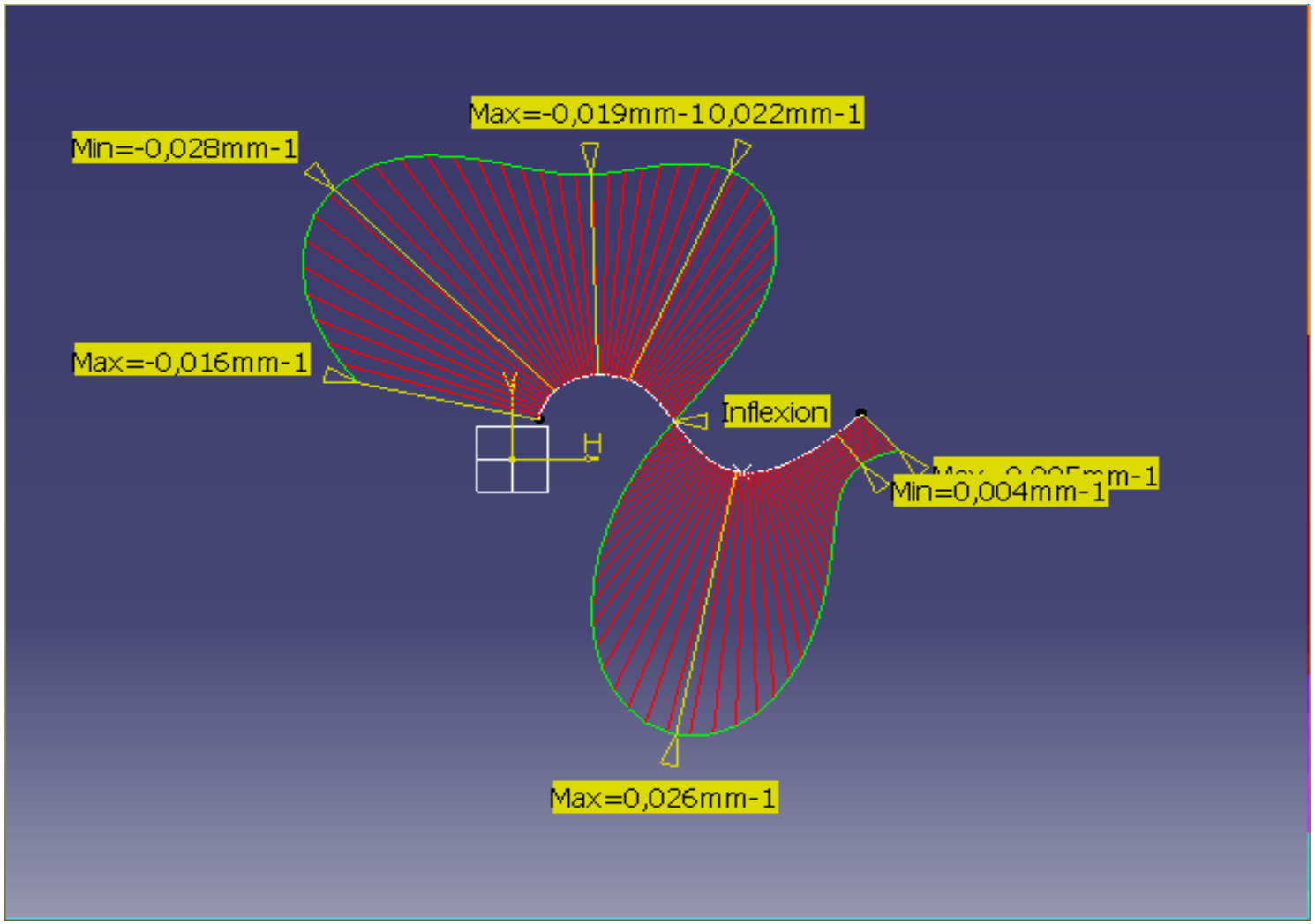
$$ds^2 = dx^2 + dy^2 + dz^2$$

$$s(u) = \int_{u_0}^{u} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

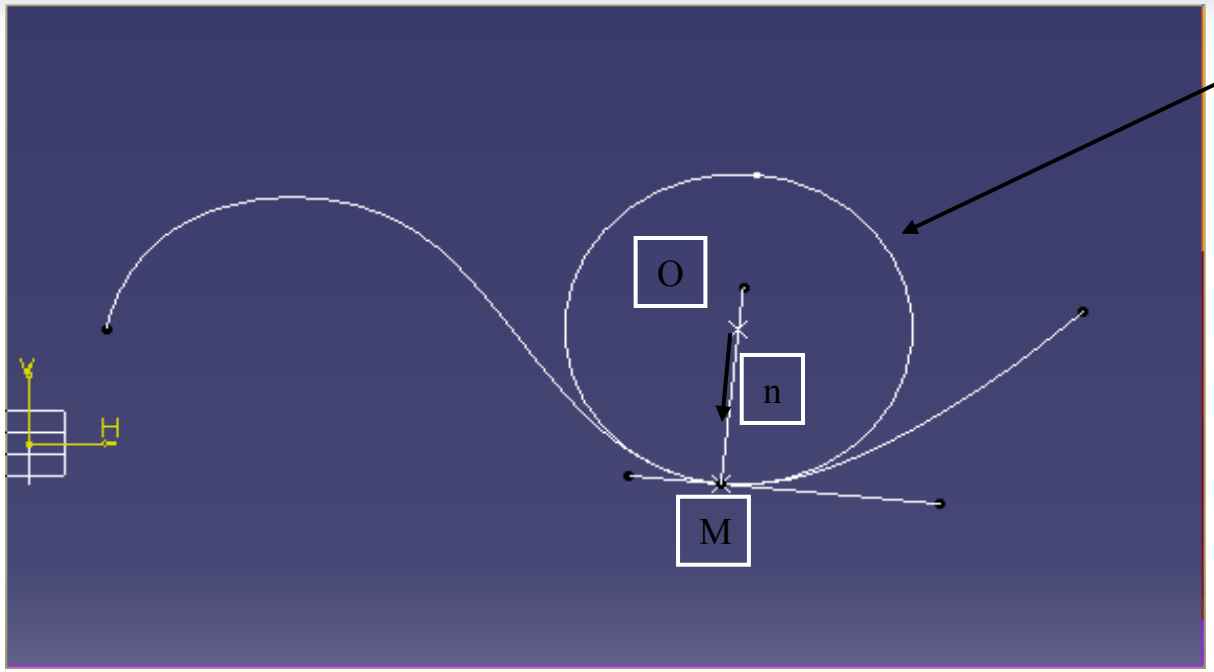
Geometry: Curvature



Geometry: Curvature



Geometry: Curvature



Ray of
Osculator Circle
=
Curvature Ray

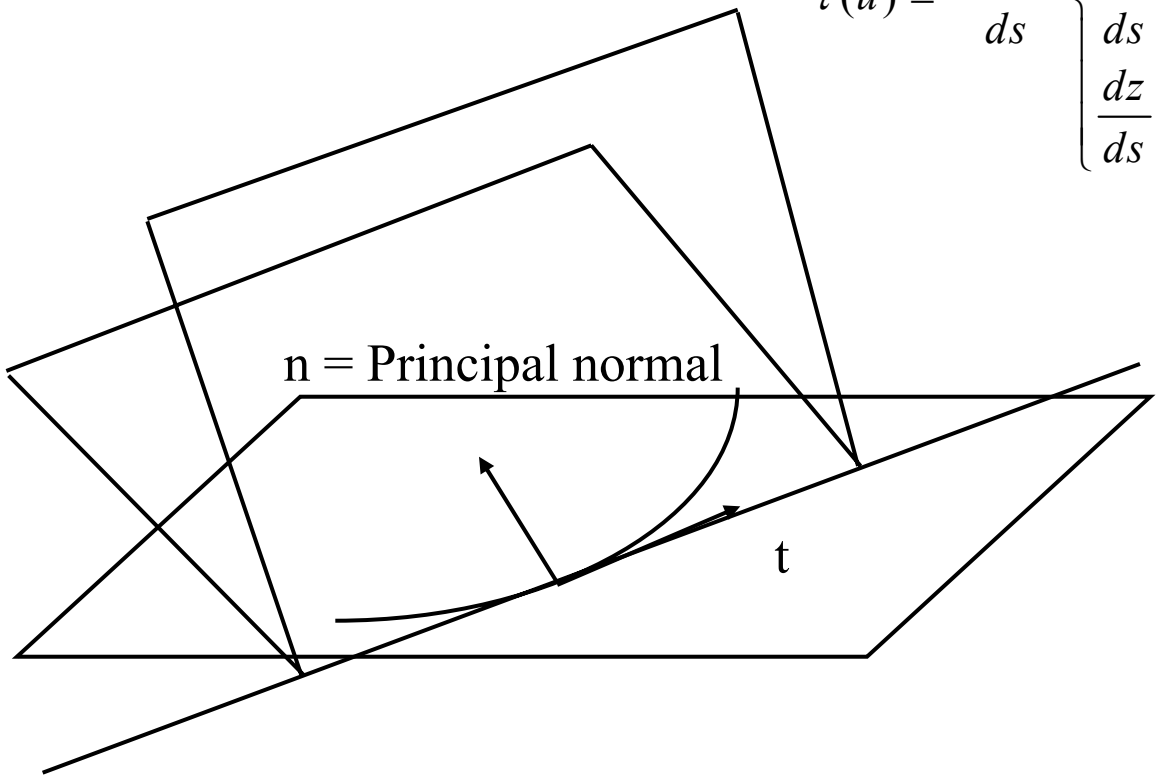
$$OM = \frac{n}{\rho}$$

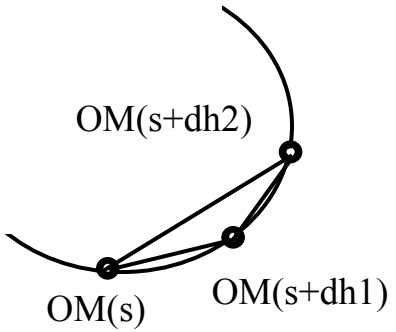
For the curve with the equation y = f(x): $R = \frac{(1 + y'^2)^{3/2}}{y''}$

Parametric curve: $R = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

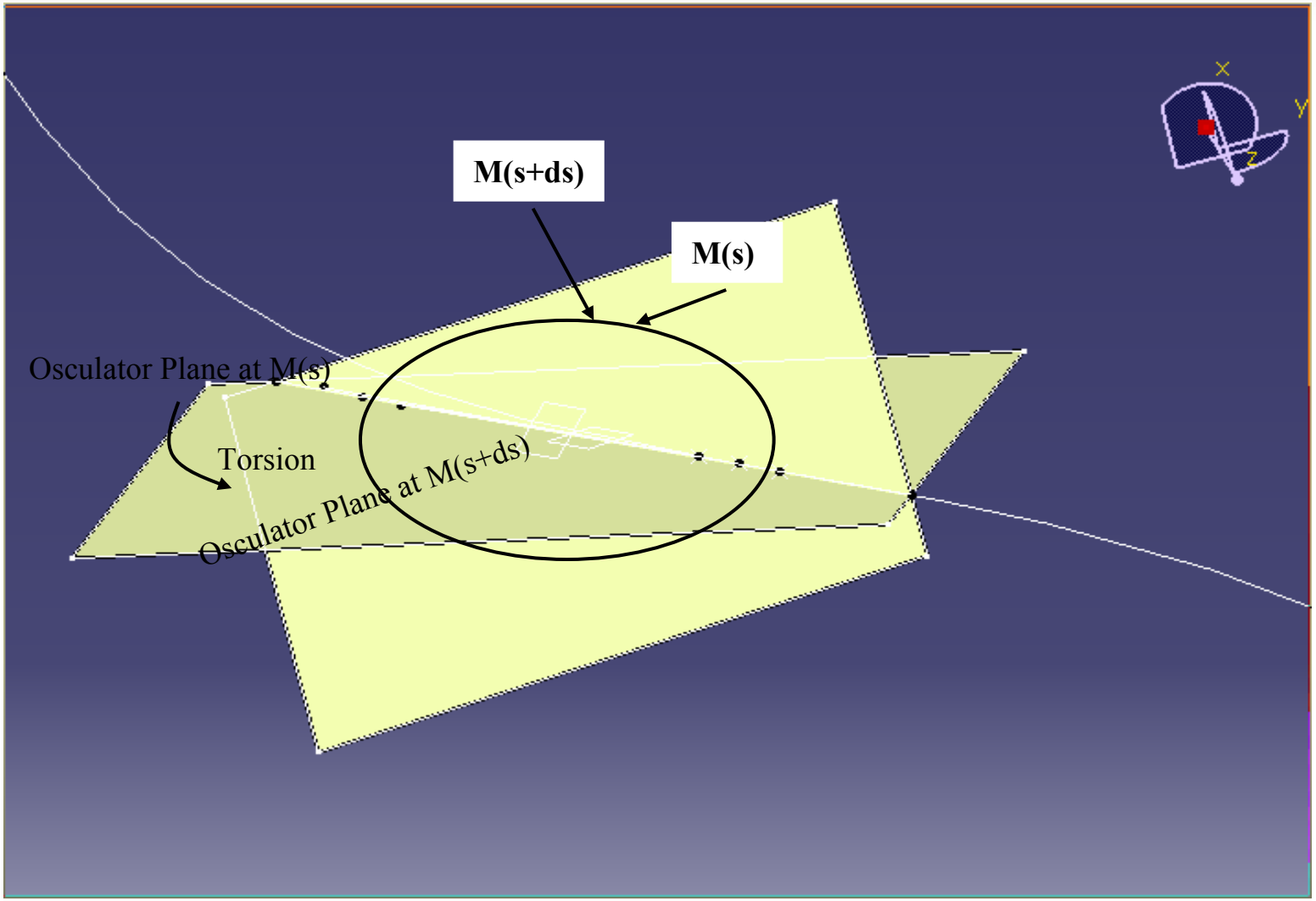
Geometry: Vector and Tangential Frenet Reference

$$\tau(u) = \frac{dOM}{ds}\begin{Bmatrix}\frac{dx}{ds}\\ \frac{dy}{ds}\\ \frac{dz}{ds}\end{Bmatrix} \quad \text{et } \mathrm{n}(u) = \frac{d^2OM}{ds^2}\begin{Bmatrix}\frac{d^2x}{ds^2}\\ \frac{d^2y}{ds^2}\\ \frac{d^2z}{ds^2}\end{Bmatrix}$$





Geometry: Torsion



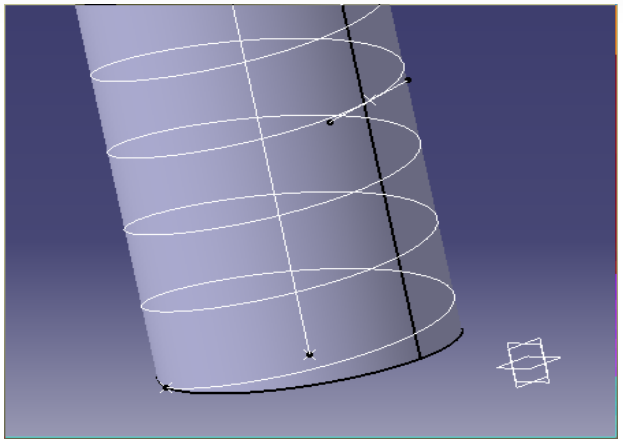
Geometry: Propeller Circular

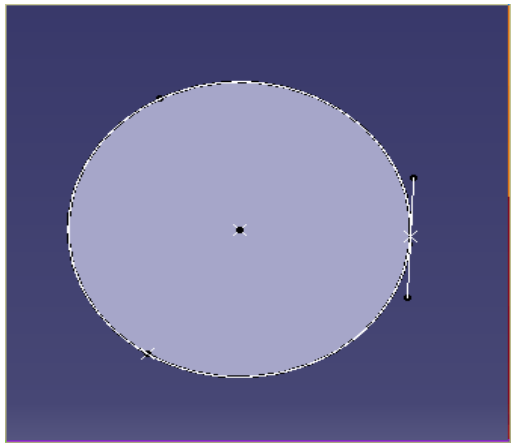
$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = p\theta \end{cases}, \begin{cases} \frac{dx}{d\theta} = -R\sin\theta \\ \frac{dy}{d\theta} = R\cos\theta \\ \frac{dz}{d\theta} = p \end{cases}, \begin{cases} \frac{d^2x}{d\theta^2} = -R\cos\theta \\ \frac{d^2y}{d\theta^2} = -R\sin\theta \\ \frac{d^2z}{d\theta^2} = 0 \end{cases}$$

ainsi $ds^2 = dx^2 + dy^2 + dz^2 = R^2 + p^2$ et donc $s = \sqrt{R^2 + p^2} \times \theta$

$$\tau \begin{cases} \frac{dx}{ds} = -\frac{R}{\sqrt{R^2 + p^2}}\sin\theta \\ \frac{dy}{ds} = \frac{R}{\sqrt{R^2 + p^2}}\cos\theta \\ \frac{dz}{ds} = \frac{p}{\sqrt{R^2 + p^2}} \end{cases} \text{ et } \frac{d\tau}{ds} \begin{cases} \frac{d^2x}{ds^2} = -\frac{R}{R^2 + p^2}\cos\theta \\ \frac{d^2y}{ds^2} = -\frac{R}{R^2 + p^2}\sin\theta \\ \frac{d^2z}{ds^2} = 0 \end{cases} \text{ avec } \rho = \frac{R}{R^2 + p^2}$$

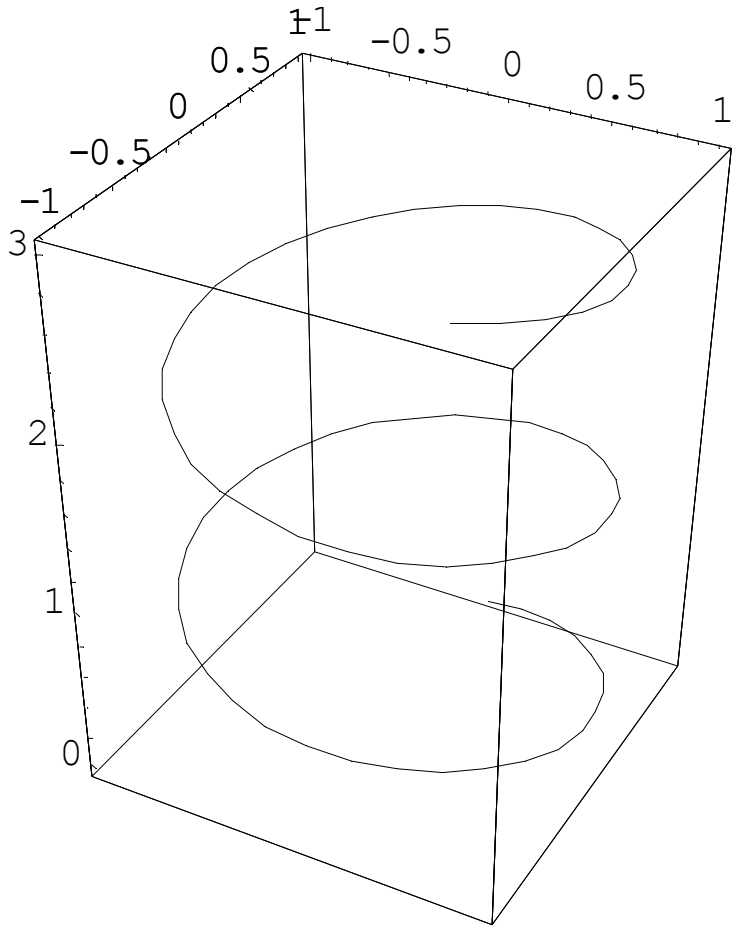
$$b = \tau \times n \begin{cases} \frac{p}{\sqrt{R^2 + p^2}}\sin\theta \\ \frac{-p}{\sqrt{R^2 + p^2}}\cos\theta \\ \frac{R}{\sqrt{R^2 + p^2}} \end{cases} \text{ et } \left\| \frac{db}{ds} \right\| = T \text{ donc } T = \frac{p}{R^2 + p^2}$$

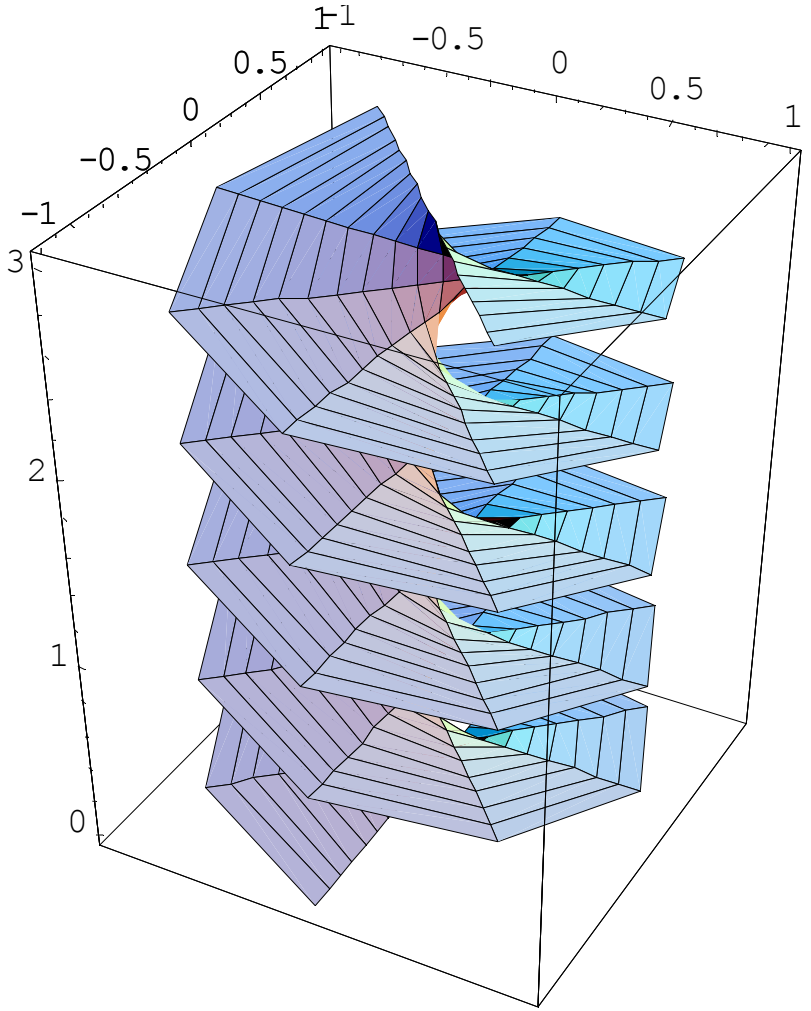




Geometry: Propeller Circular

Discretizing





Geometry: Frenet - Serret Reference, Equations, Curvature, Torsion

The Frenet - Serret equations are a convenient framework for analyzing curvature.
T(s) is the unit tangent to the curve as a function of path length s.
N(s) is the unit normal to the curve
B(s) is the unit binormal; the vector cross product of T(s) and N(s).

Frenet Reference: $B = T \wedge N$

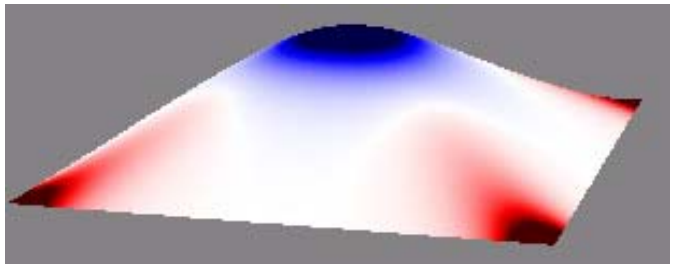
Frenet Equations:
$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

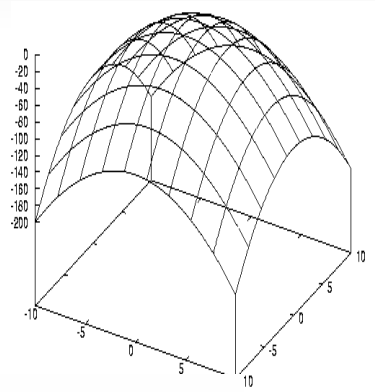
For any parametric function *f(t),* the expression of the curvature and the torsion are the following:

$$\gamma = \frac{\|f' \wedge f''\|}{\|f'\|^3} \qquad \tau = -\frac{[f', f'', f''']}{\|f' \wedge f''\|^2}$$

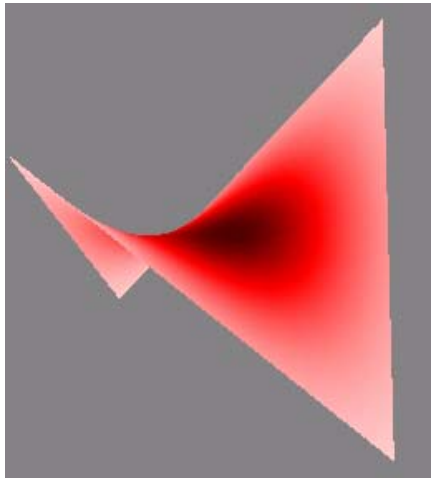
Geometry: Curvature
Gaussian, Average

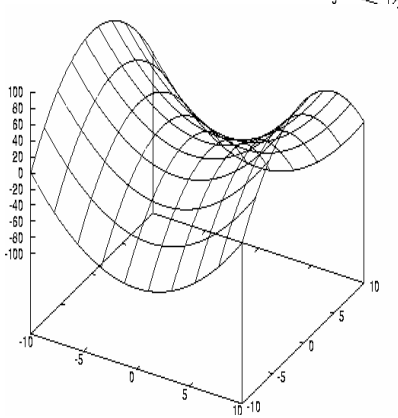
Gaussian Curvature



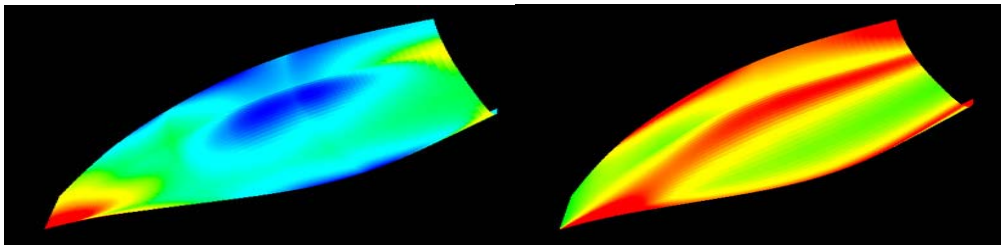


Average Curvature



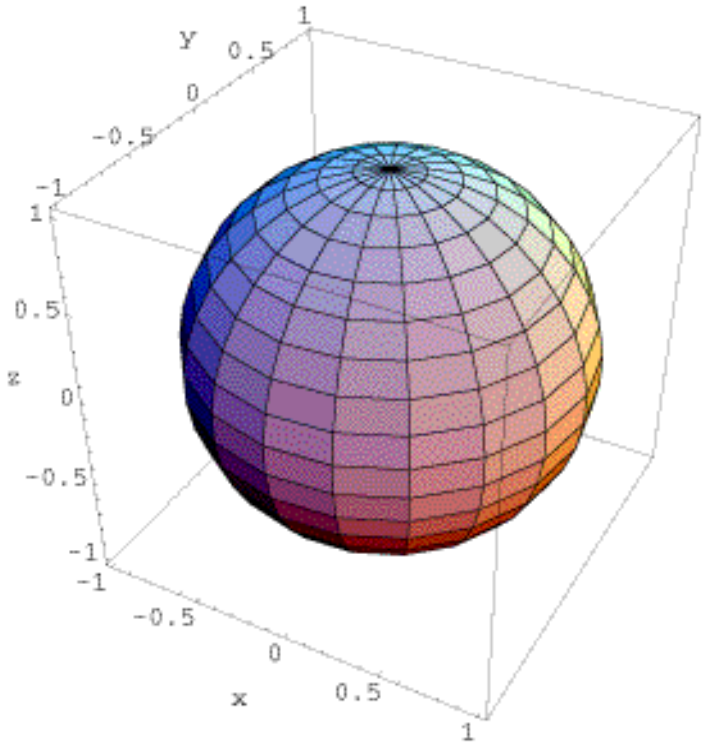


Curvature Cmap



Geometry: Curvature Gaussian, Average

Sphere

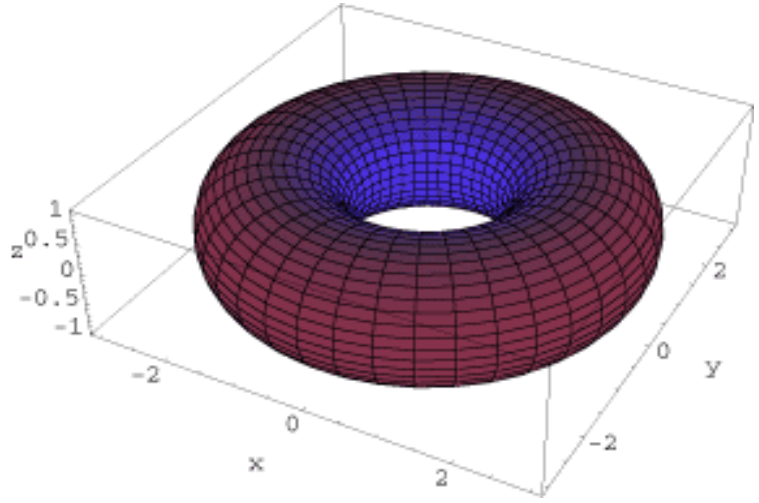


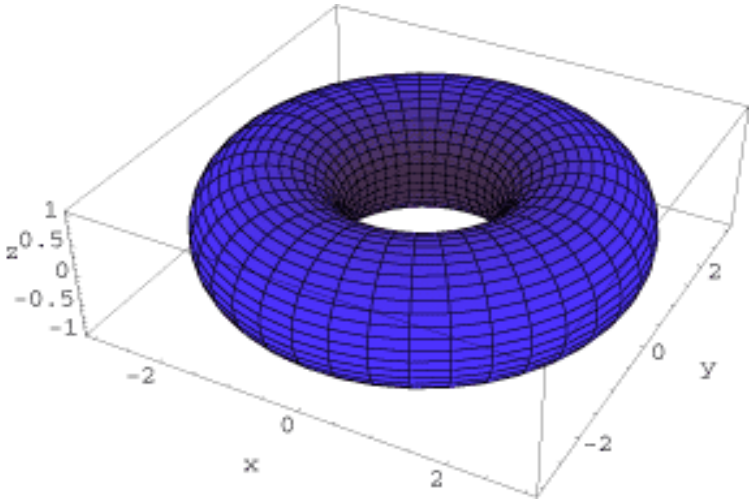
$$\text{Gaussian Curvature} = \frac{1}{R^2}$$

$$\text{Average Curvature} = -\frac{1}{R}$$

Geometry: Curvature Gaussian, Average

Torus





Gaussian

$$K[u, v] = \frac{\text{Cos}[v]}{b\ (a + b\ \text{Cos}[v])}$$

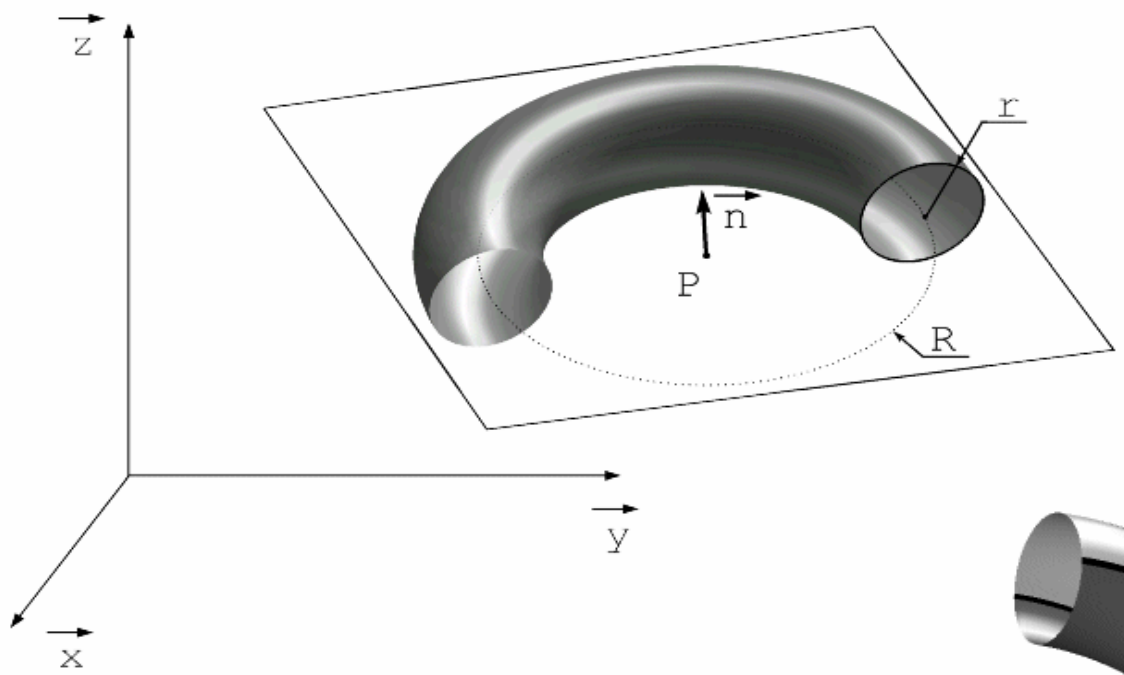
Average

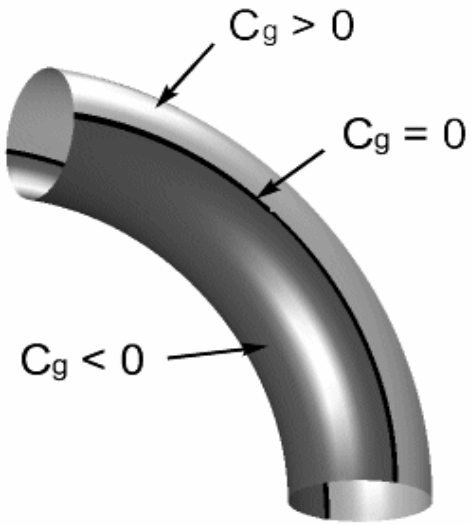
$$H[u, v] = -\frac{a + 2\ b\ \text{Cos}[v]}{2\ b\ (a + b\ \text{Cos}[v])}$$

$$\begin{pmatrix} x[u, v] \\ y[u, v] \\ z[u, v] \end{pmatrix} = \begin{pmatrix} \text{Cos}[u]\ (a + b\ \text{Cos}[v]) \\ (a + b\ \text{Cos}[v])\ \text{Sin}[u] \\ b\ \text{Sin}[v] \end{pmatrix}$$

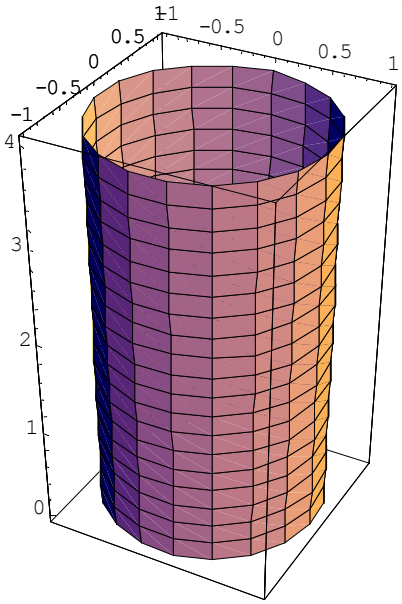
Geometry: Curvature

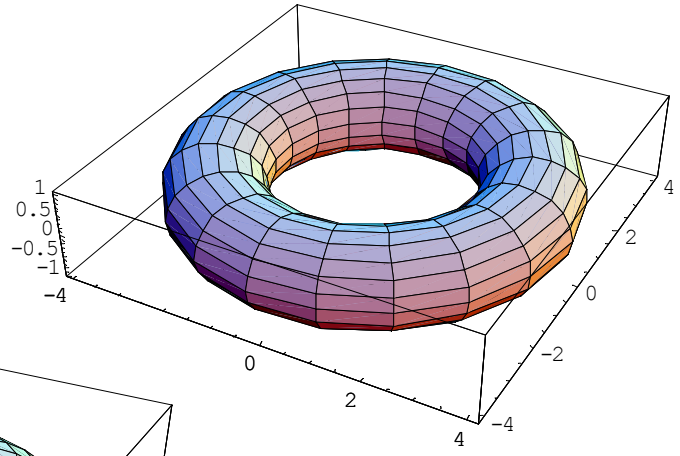
Torus

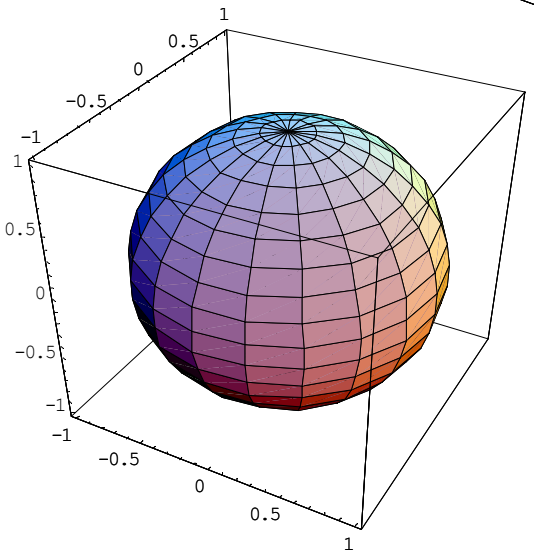




Geometry: Surfaces of Revolution



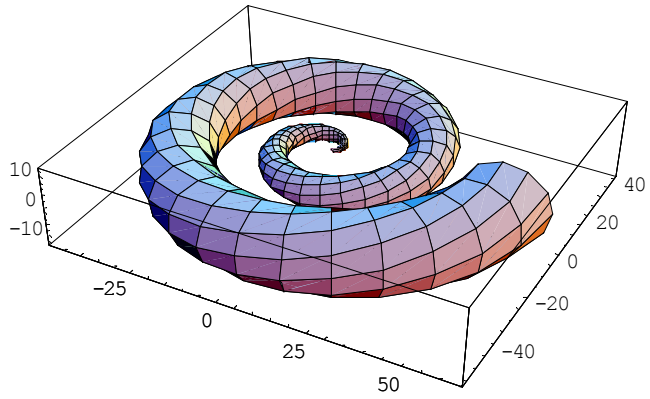


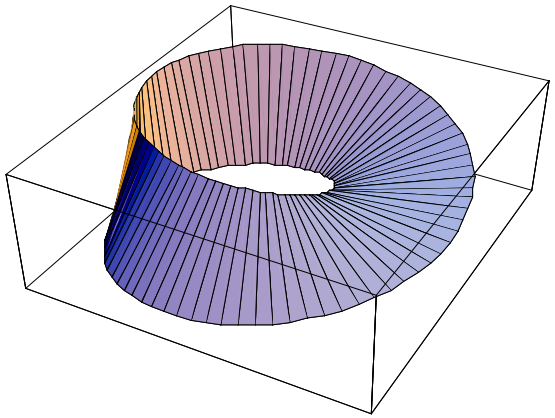


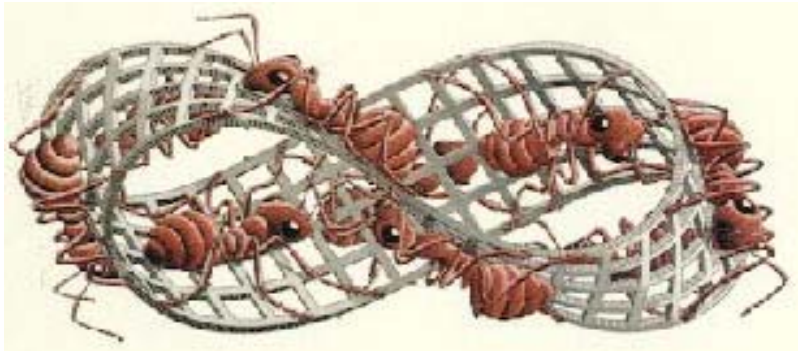
Geometry: Particular Surfaces

Mobius Strip: 'Endless Ribbon'

$$\begin{cases} x = \left(R + u\cos\dfrac{v}{2}\right)\cos v \\ y = \left(R + u\cos\dfrac{v}{2}\right)\sin v \\ z = u\sin\dfrac{v}{2} \end{cases}$$

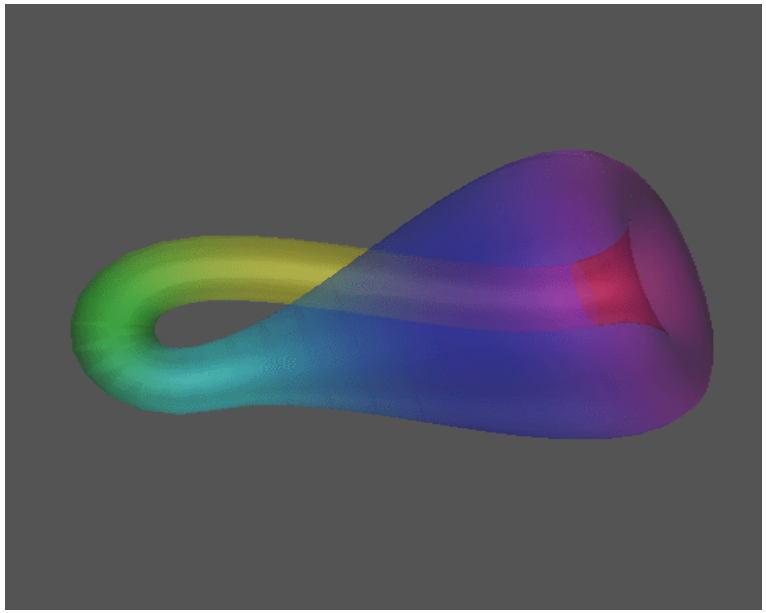


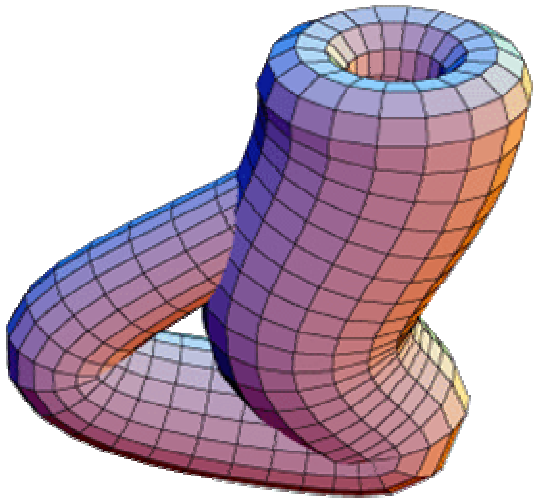




Geometry: Particular Surfaces

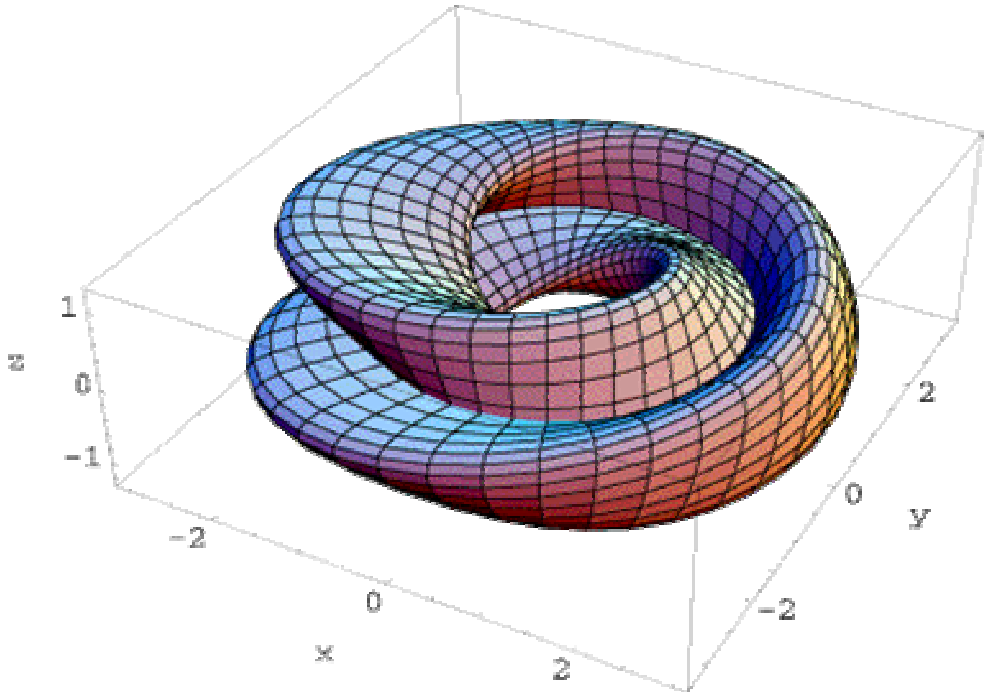
The Klein Bottle

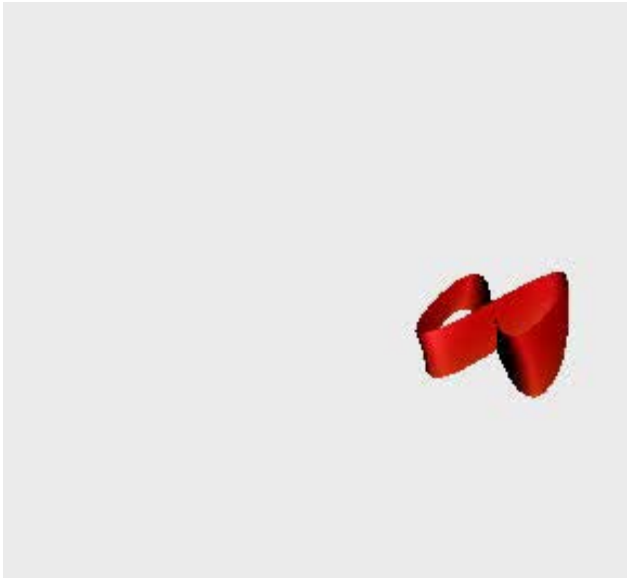






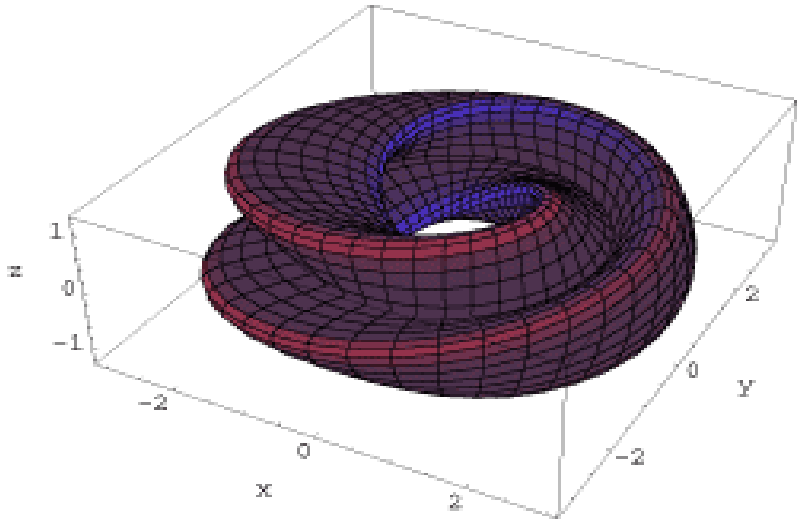
The Klein Bottle

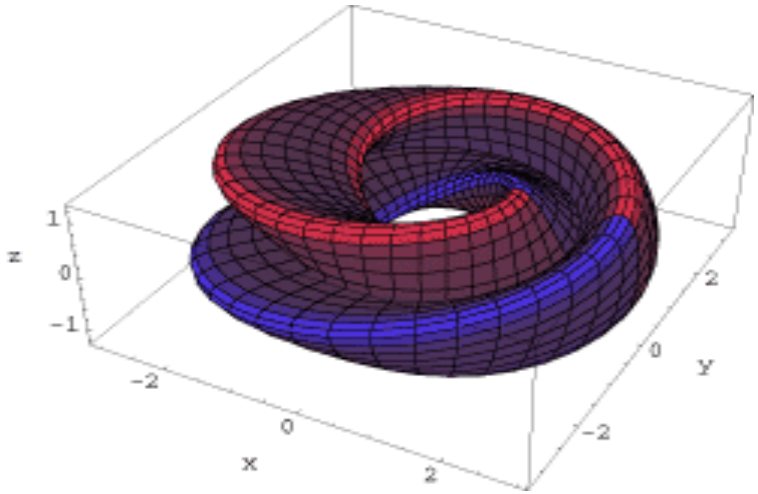


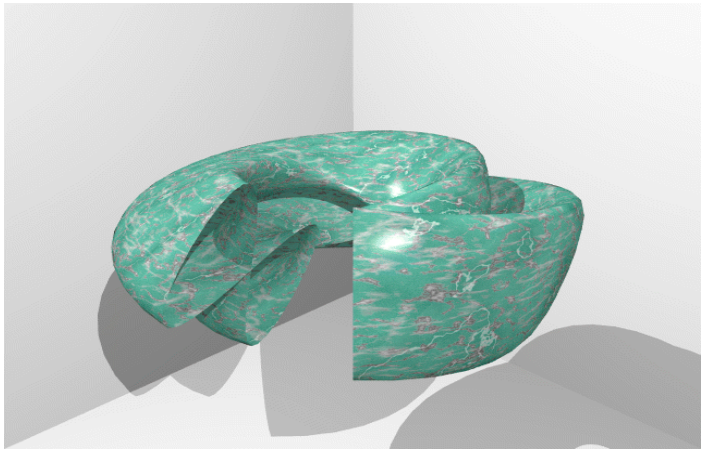


$$\begin{pmatrix} x[u, v] \\ y[u, v] \\ z[u, v] \end{pmatrix} = \begin{pmatrix} \text{Cos}[u]\ (a + \text{Cos}[\frac{u}{2}]\ \text{Sin}[v] - \text{Sin}[\frac{u}{2}]\ \text{Sin}[2\,v]) \\ \text{Sin}[u]\ (a + \text{Cos}[\frac{u}{2}]\ \text{Sin}[v] - \text{Sin}[\frac{u}{2}]\ \text{Sin}[2\,v]) \\ \text{Sin}[\frac{u}{2}]\ \text{Sin}[v] + \text{Cos}[\frac{u}{2}]\ \text{Sin}[2\,v] \end{pmatrix}$$

The Klein Bottle: Curvature

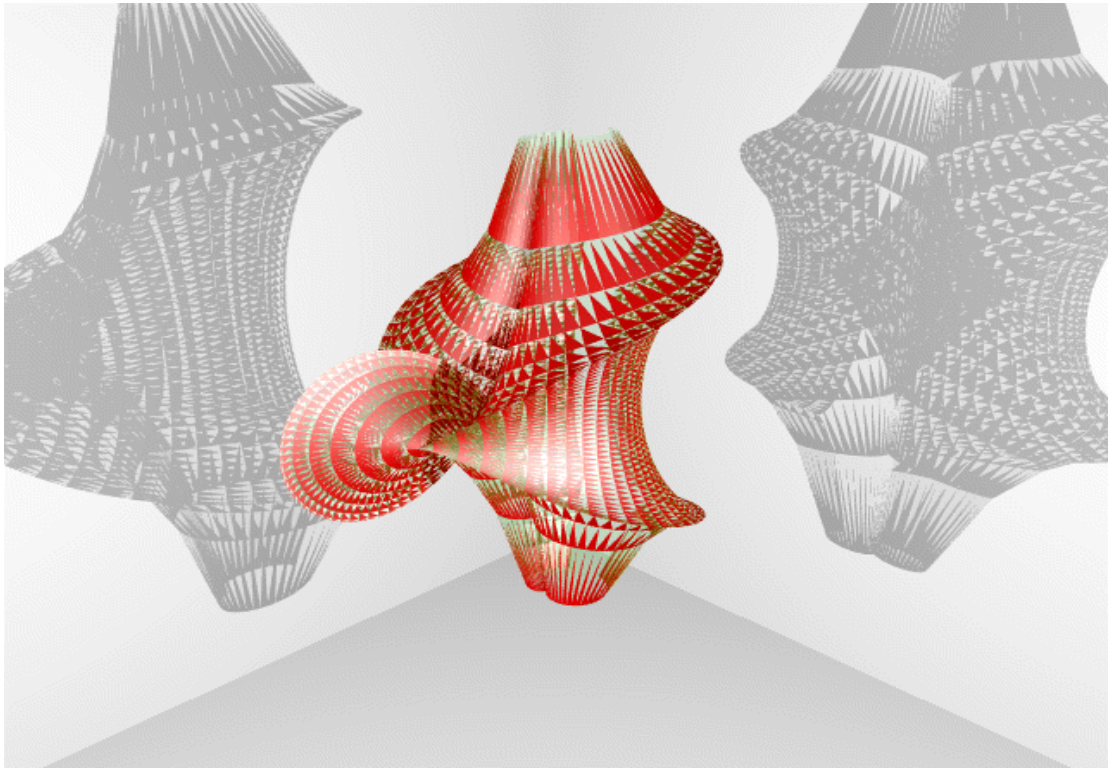






Geometry: Particular Surfaces

The Kuen Surfaces

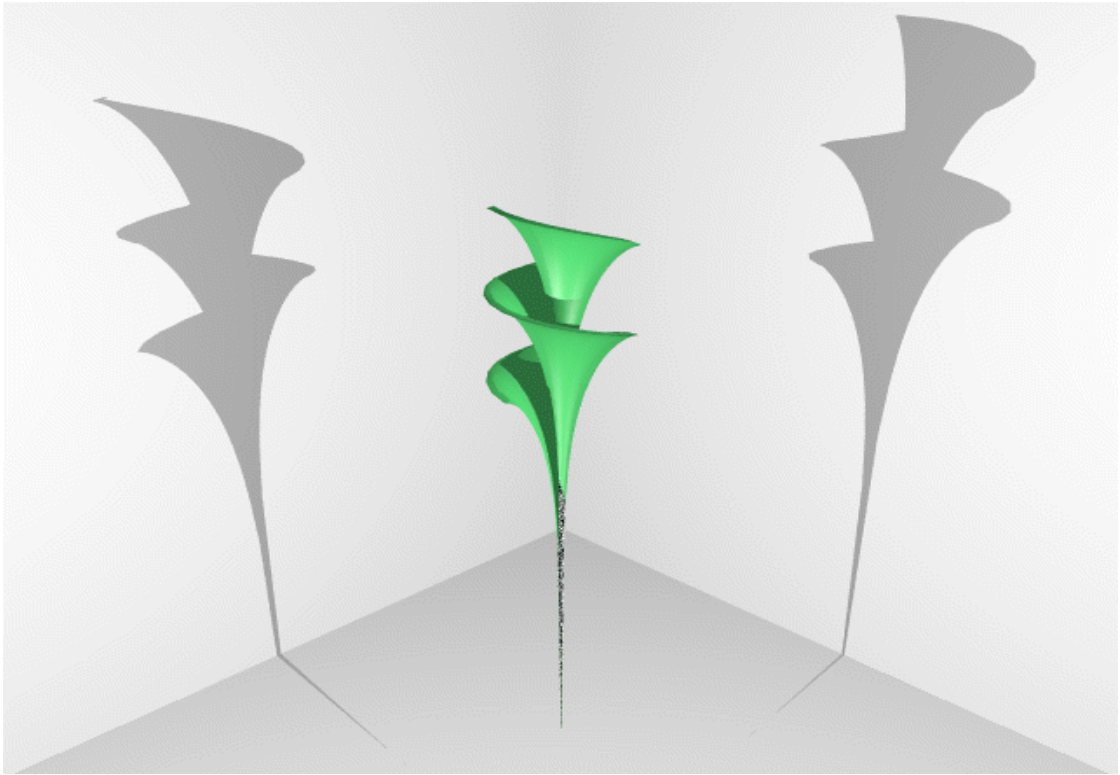


$$x=2*(\cos(u)+u*\sin(u))*\sin(v)/(1+u*u*\sin(v)*\sin(v))$$

$$y=2*(\sin(u)-u*\cos(u))*\sin(v)/(1+u*u*\sin(v)*\sin(v))$$

$$z=\log(\tan(v/2))+2*\cos(v)/(1+u*u*\sin(v)*\sin(v))$$

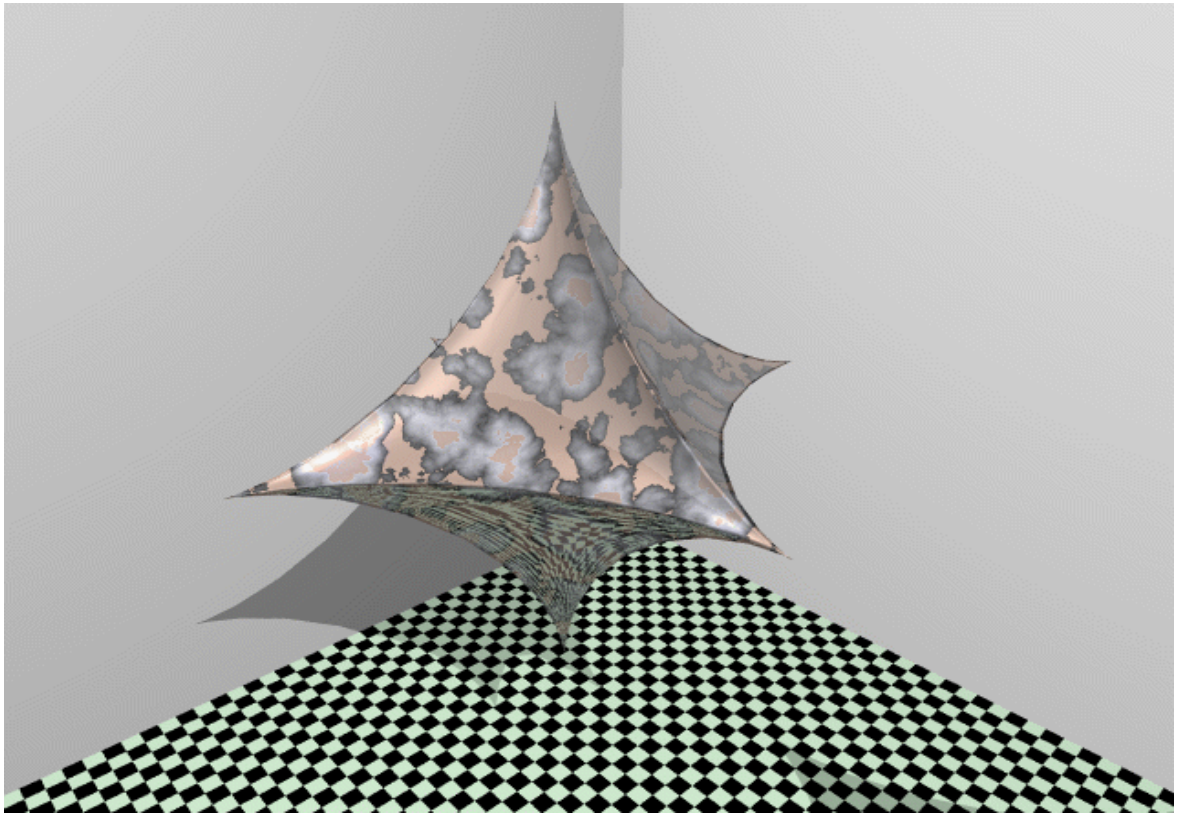
The Dini Surfaces



x=a*cos(u)*sin(v)
y=a*sin(u)*sin(v)
z=a*(cos(v)+log(tan((v/2))))+b*u
a=1,b=0.2,u={ 0,4*pi},v={0.001,2}

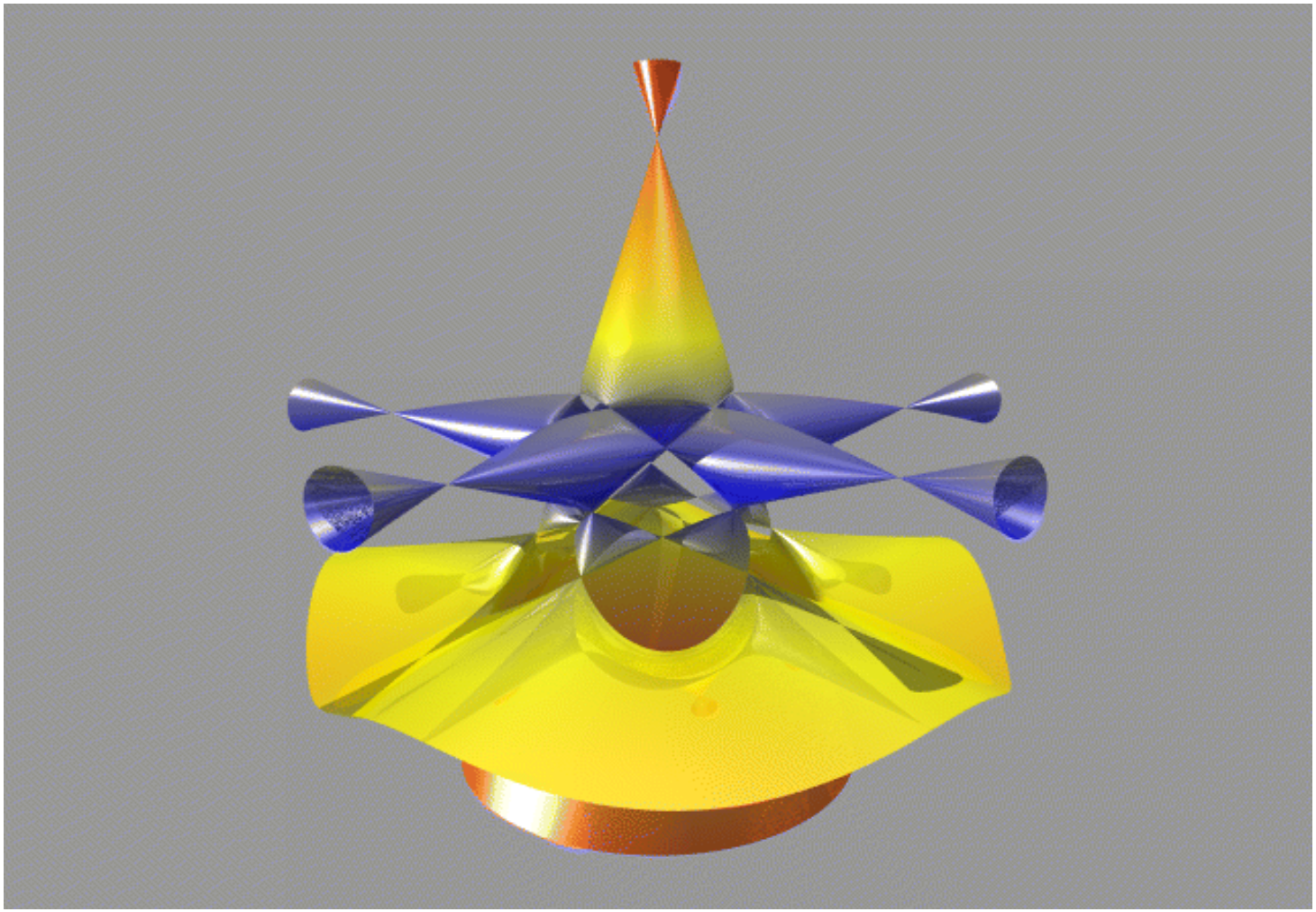
Geometry: Particular Surfaces

Asteroid

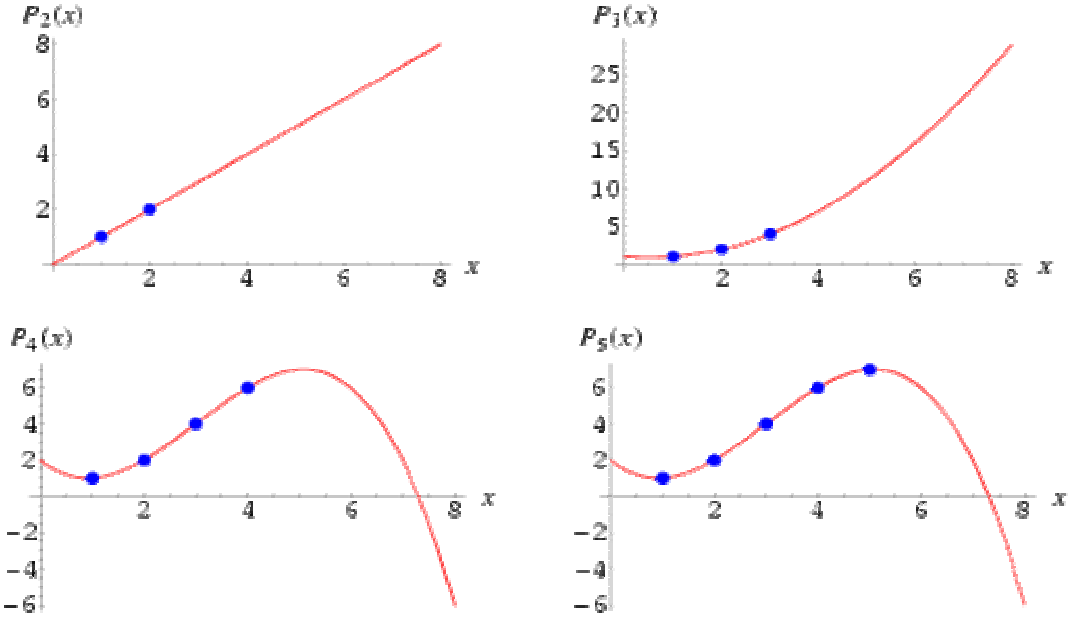


x= pow (a*cos(u)*cos(v),3)
y= pow (b*sin(u)*cos(v),3)
z= pow (c*sin(v),3)

The «Derviche»



Lagrange Interpolating Polynomial



The Lagrange interpolating polynomial is the polynomial *P(x)* of degree <= (n-1) that passes through the n points (x_1,y_1= f(x_1)), x_2,y_2= f(x_2)), ..., x_n,y_n= f(x_n)), and is given by:

$$P(x) = \sum_{j=1}^{n} P_j(x),$$

Lagrange Interpolating Polynomial

$$P(x) = \sum_{j=1}^{n} P_j(x),$$

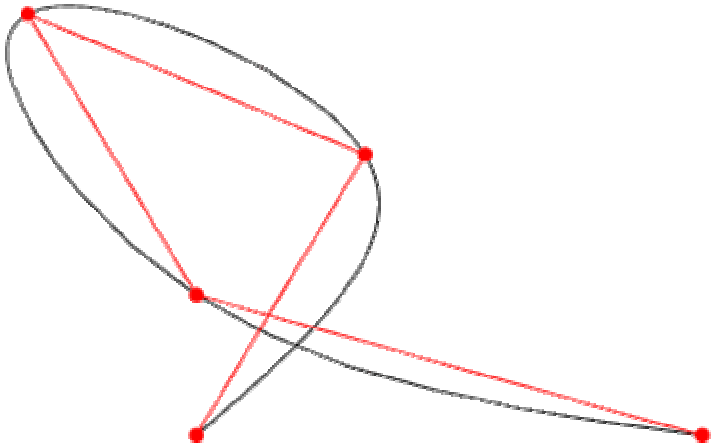
Where:

$$P_j(x) = y_j \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k},$$

Written explicitly:

$$P(x) = \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2 + \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} y_n.$$

Cubic Spline Interpolating Polynomial



A cubic spline is a spline constructed of piecewise third-order polynomials which pass through a set of *m* control points.
The second derivative of each polynomial is commonly set to zero at the endpoints, since this provides a boundary condition that completes the system of *m -2* equations.
This produces a so-called "natural" cubic spline and leads to a simple tridiagonal system which can be solved easily to give the coefficients of the polynomials.
However, this choice is not the only one possible, and other boundary conditions can be used instead.

Cubic Spline Interpolating Polynomial

Consider 1-dimensional spline for a set of *n+1* points (y_1, y_2, .., y_n), let the *i*th piece of the spline be represented by:

$$Y_i(t) = a_i + b_i t + c_i t^2 + d_i t^3,$$

Where *t* is a parameter $t \in [0, 1]$ and $i = 0, \ldots, n\text{-}1$ then

Rearranging all these equations, leads to the following beautifully symmetric tridiagonal system:

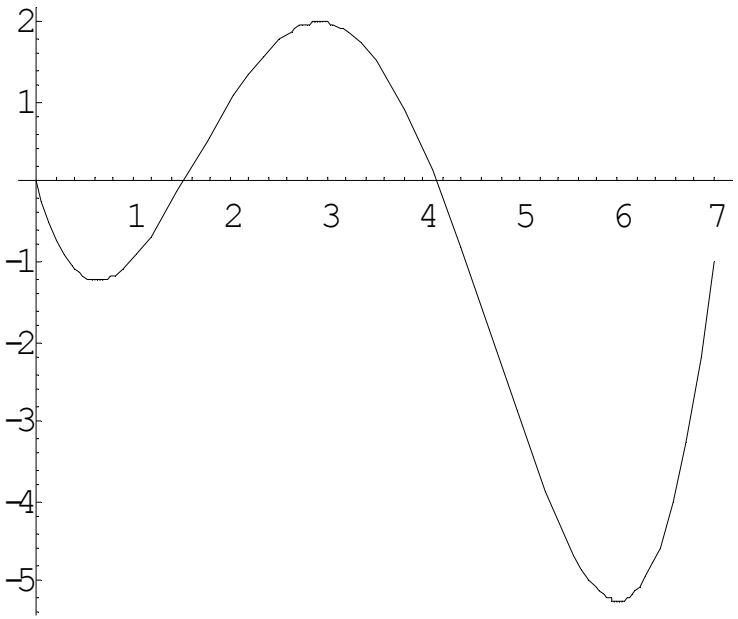
$$\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} = \begin{bmatrix} 3(y_1 - y_0) \\ 3(y_2 - y_0) \\ 3(y_3 - y_1) \\ \vdots \\ 3(y_{n-1} - y_{n-3}) \\ 3(y_n - y_{n-2}) \\ 3(y_n - y_{n-1}) \end{bmatrix}.$$

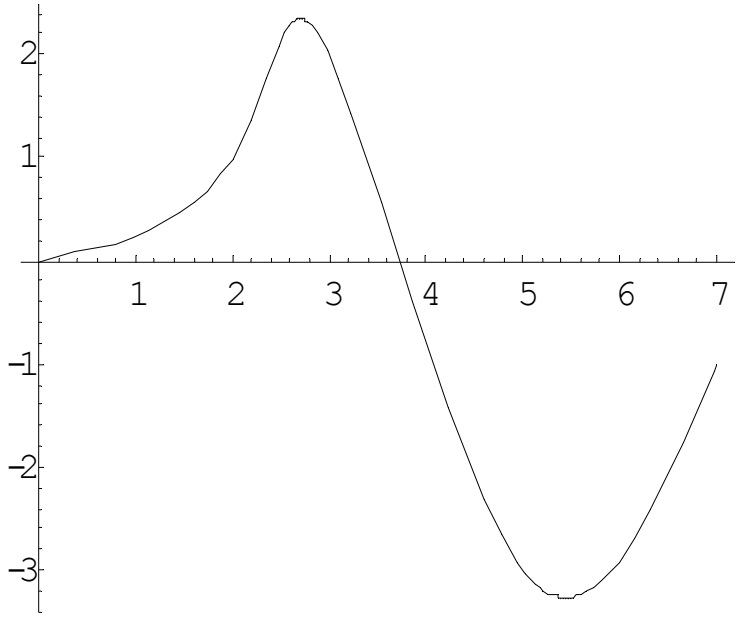
If the curve is instead closed, the system becomes

$$\begin{bmatrix} 4 & 1 & & & & & 1 \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & 1 & 4 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & 4 & 1 \\ 1 & & & & & 1 & 4 \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{n-1} \\ D_n \end{bmatrix} = \begin{bmatrix} 3(y_1 - y_n) \\ 3(y_2 - y_0) \\ 3(y_3 - y_1) \\ \vdots \\ 3(y_{n-1} - y_{n-3}) \\ 3(y_n - y_{n-2}) \\ 3(y_0 - y_{n-1}) \end{bmatrix}.$$

Geometry: Curves

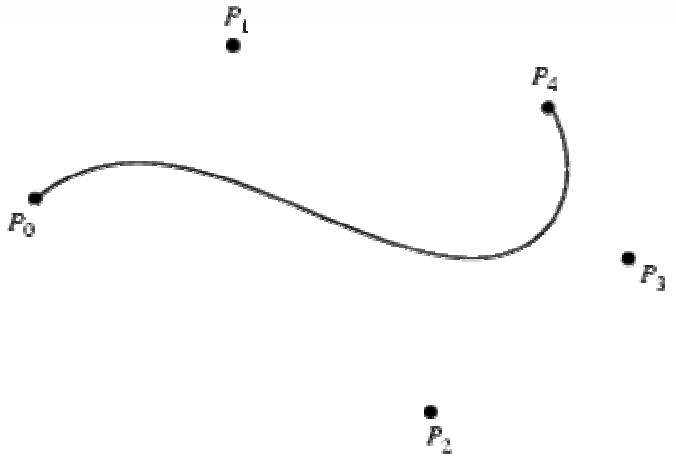
Cubic Spline/Lagrange





Geometry: Curves

Bézier



Given a set of $n + 1$ control points $P_0, P_1, .., P_n$, the corresponding Bèzier curve (or Bernstein- Bèzier curve) is given by:

$$\mathbf{C}(t) = \sum_{i=0}^{n} \mathbf{P_i} B_{i,n}(t),$$

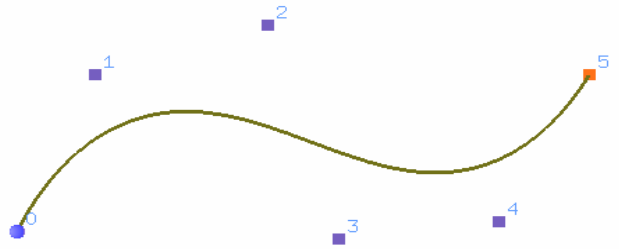
Where $B_{i,n}(t)$ is a Bernstein polynomial and $t \in [0, 1]$.

A "rational" Bézier curve is defined by: $\mathbf{C}(t) = \dfrac{\sum_{i=0}^{n} B_{i,p}(t) w_i \mathbf{P_i}}{\sum_{i=0}^{n} B_{i,p}(t) w_i}$,

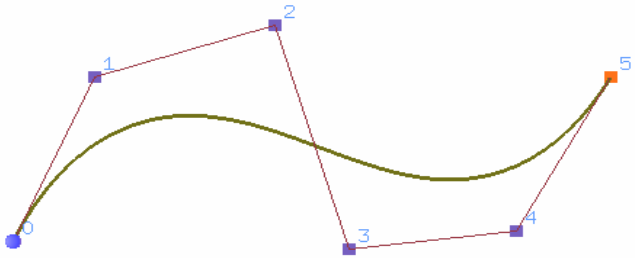
where p is the order, $B_{i,p}$ are the Bernstein polynomials, $\mathbf{P_i}$ are control points, and the weight W_i of $\mathbf{P_i}$ is the last ordinate of the homogeneous Point $\mathbf{P_i}^w$. These curves are closed under perspective transformations, and can represent conic sections exactly.

Bézier:Properties

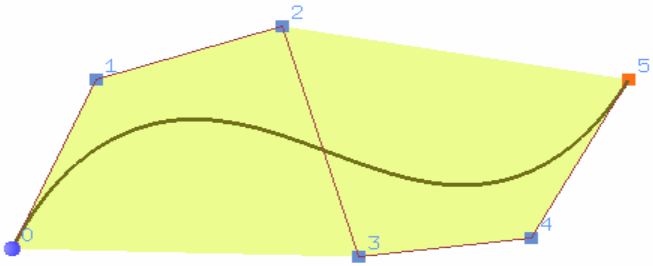
- The Bézier curve always passes through the first and last control points.



- The curve is tangent to P1–P0 and Pn–Pn-1 at the endpoints.

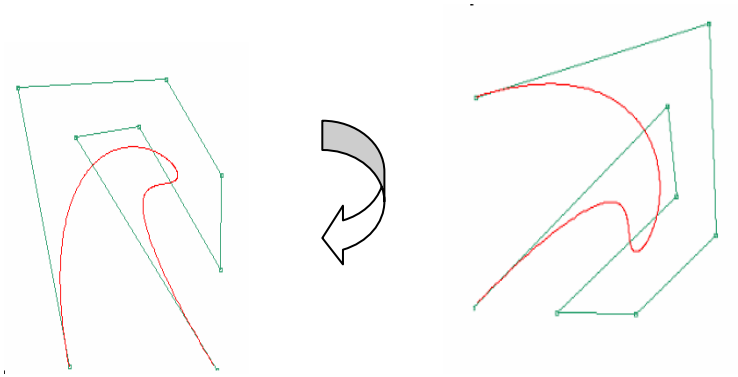


- The curve lies within the convex hull of the control points.

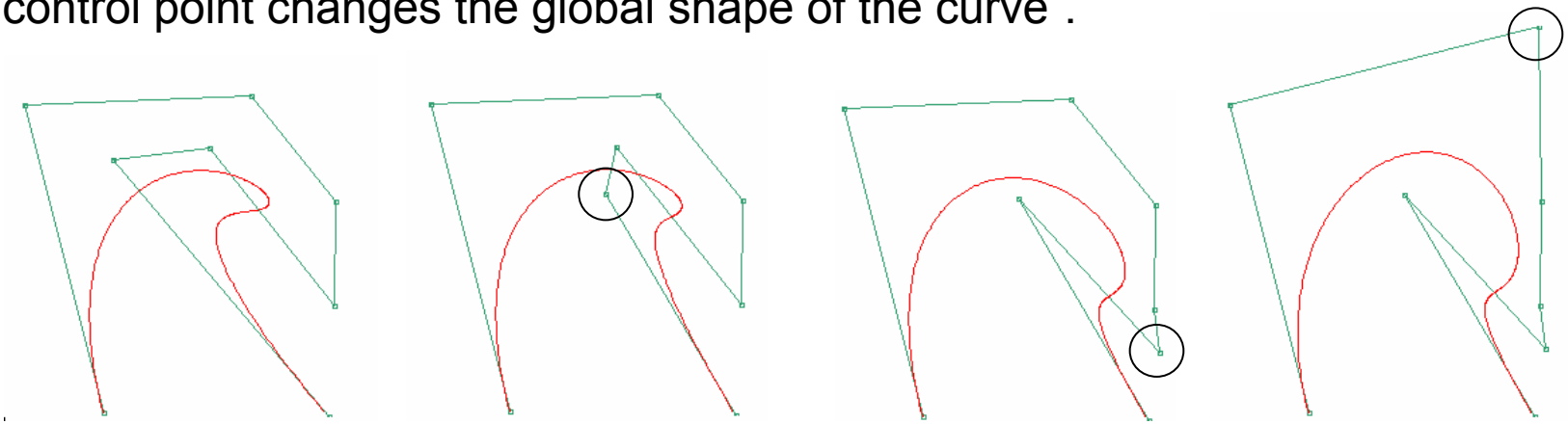


Bézier:Properties

- A desirable property is that the curve can be translated and rotated by performing these Operations on the control points.

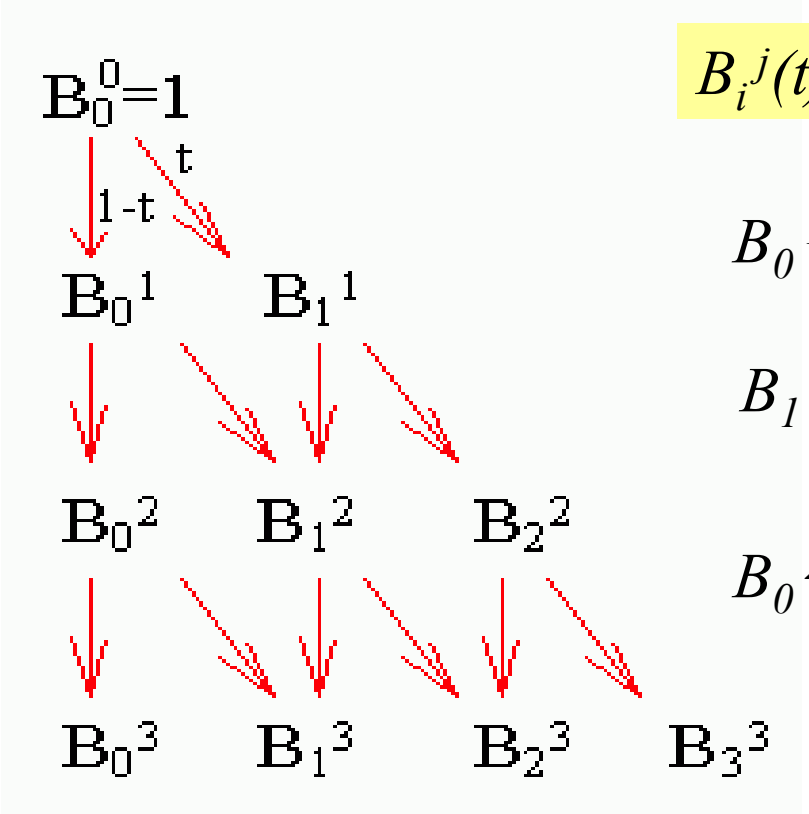


- Undesirable properties of Bézier curves are their numerical instability for large numbers of control points, and the fact that moving a single control point changes the global shape of the curve .



Bézier: Bernstein Polynomials

$$B_i^n(t) = C_n^i\,(t\text{-}1)^{n-i}t^i\ ,\ C_n^i = n!\,/\,i!(n-i)!$$



$$B_i^j(t) = (1\text{-}t)B_i^{j-1}(t) + t\,B_{i-1}^{j-1}(t)$$

$$B_0^1(t) = (1\text{-}t)B_0^0(t) + t\,B_{-1}^0(t)$$

$$B_1^1(t) = (1\text{-}t)B_1^0(t) + t\,B_0^0(t)$$

$$B_0^2(t) = (1\text{-}t)B_0^1(t) + t\,B_{-1}^1(t)$$

Bézier: Bernstein Polynomials

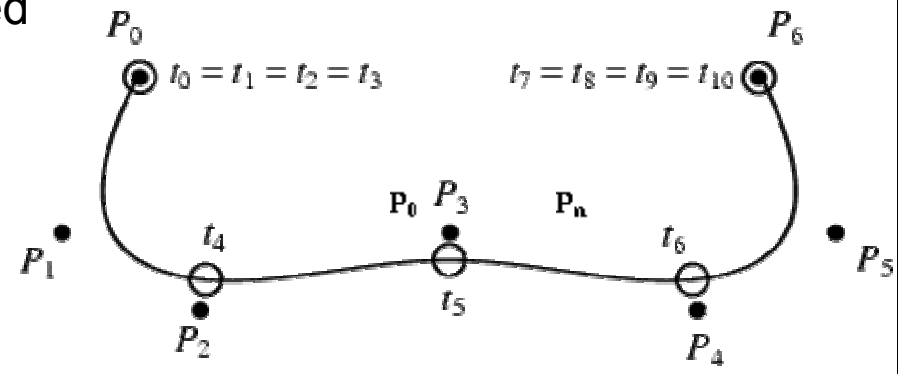
$$B_i^j(t) = (1-t)B_i^{j-1}(t) + t\,B_{i-1}^{j-1}(t)$$

- Unit Partition: $\Sigma_{i=0,n}\, B_i^n(t) = 1$
- $0 <= B_i^n(t) <= 1$
- $B_i^n(0) = 0$ et $B_i^n(1) = 0$
- $B_0^n(0) = 1$
- $B_n^n(1) = 1$

B-Spline

A B-Spline is a generalization of the Bézier curve.
Let a vector known as the knot vector be defined

$$T = \{t_0, t_1, \ldots, t_m\},$$



where T is a no decreasing sequence with $t \in [0, 1]$
and define control points P_0, ..., P_n.
Define the degree as: $p = m-n-1$

The "knots" t_{p+1}, ..., t_{m-p-1} are called internal knots.

B-Spline

- Define the basis functions as:

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \le t < t_{i+1} \text{ and } t_i < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t).$$

- Then the curve defined by: $\mathbf{C}(t) = \sum_{i=0}^{n} \mathbf{P_i} N_{i,p}(t)$ is a B-spline.

- Specific types include the non periodic B-spline (first *p+1* knots equal 0 and last *p+1* equal to 1; illustrated above) and uniform B-spline (internal knots are equally spaced).

- A B-spline with no internal knots is a Bézier curve.

- A curve is *p* - *k* times differentiable at a point where *k* duplicate knot values occur.

NURBS-Curve

A non uniform rational B-spline curve defined by:

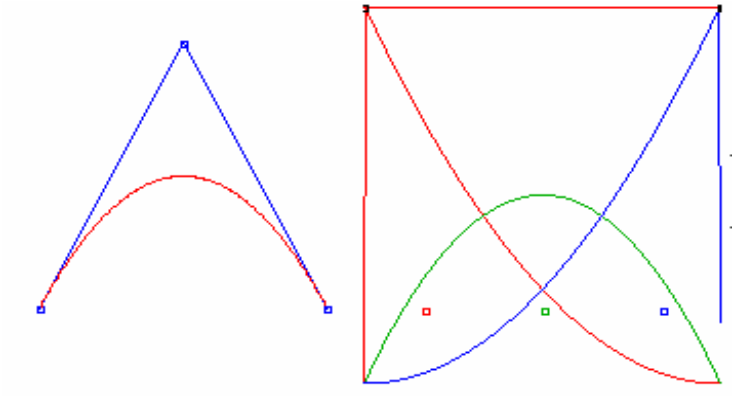
$$\mathbf{C}(t) = \frac{\sum_{i=0}^{n} N_{i,p}(t)\, w_i\, \mathbf{P_i}}{\sum_{i=0}^{n} N_{i,p}(t)\, w_i},$$

where p is the order, $N_{i,p}$ are the B-Spline basis functions, $\mathbf{P_i}$ are control points, and the weight W_i of $\mathbf{P_i}$ is the last ordinate of the homogeneous point $\mathbf{P_i}^w$. These curves are closed under perspective transformations, and can represent conic sections exactly.

Geometry: Curves

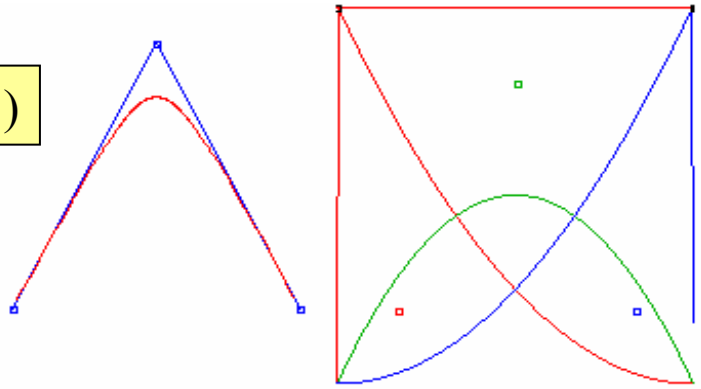
Conics

$$P(t) = \frac{\sum_{i=0}^{i=3} w_i \times N_{i,2}(t) P_i(t)}{\sum_{i=0}^{i=3} w_i \times N_{i,2}(t)}$$

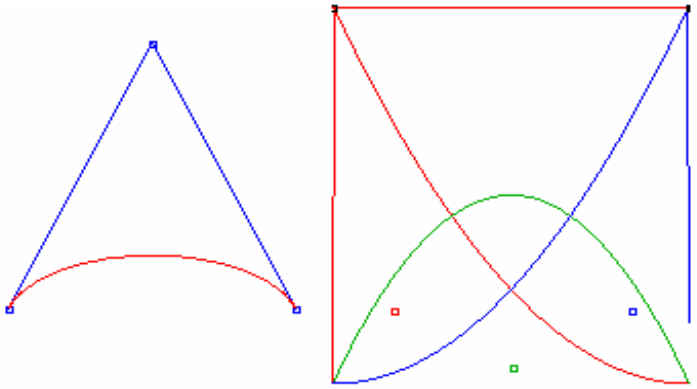


Parabola w=1

(0,0,0,1,1,1)



Hyperbole w=4

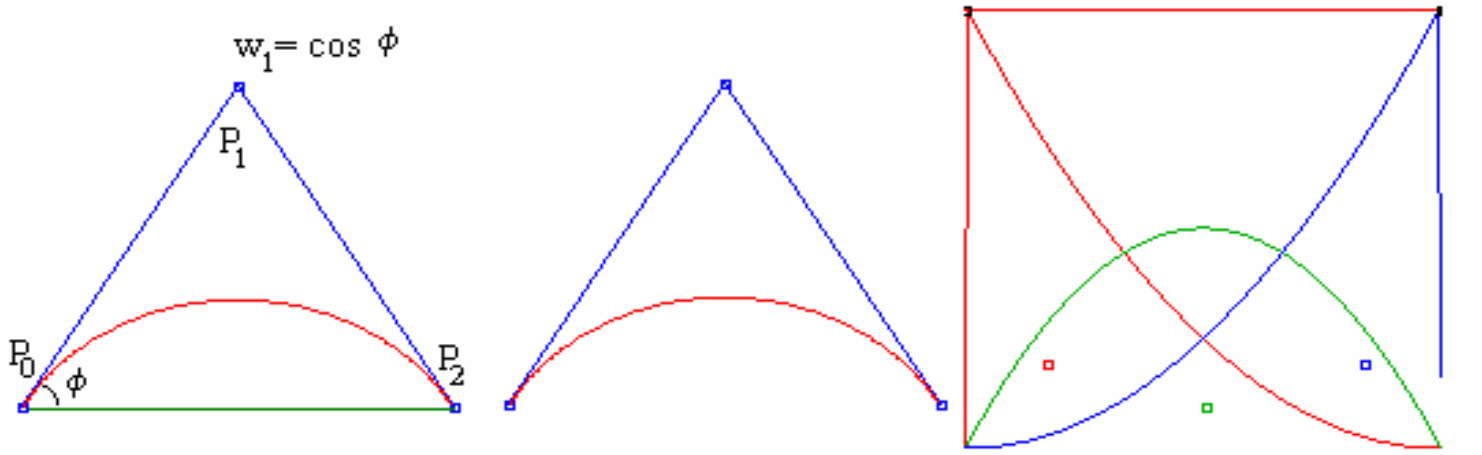


Ellipse w=1/4

Geometry: Curves

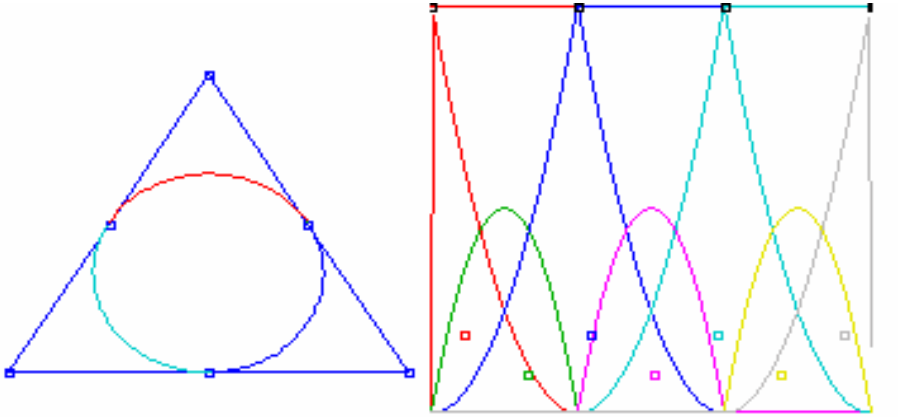
Arcs

- NURBS of degree 2
- Control points (Isosceles triangle)
- Knot vector (0,0,0,1,1,1)

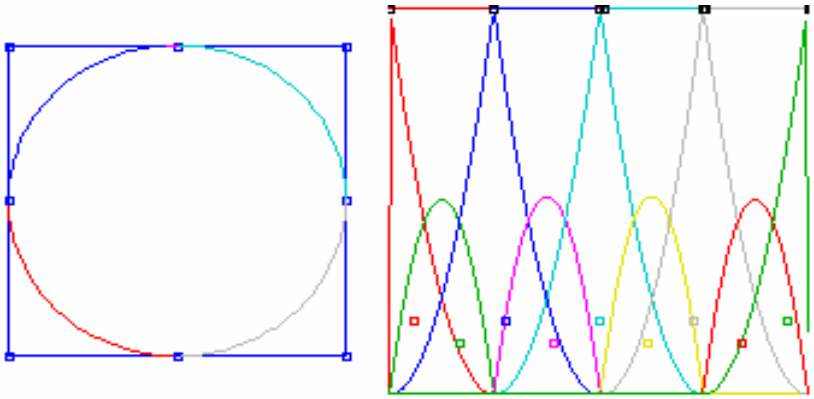


Geometry: Curves

Circles



(0, 0, 0, 1/3, 1/3, 2/3, 2/3, 1, 1, 1)



(0, 0, 0, 1/4, 1/4, 1/2, 1/2, 3/4, 3/4, 1, 1, 1)

Bézier

A given Bézier surface of order (*n*, *m*) is defined by a set of (*n* + 1)(*m* + 1) control points **k***i,j*.

$$\mathbf{p}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_i^n(u)\, B_j^m(v)\, \mathbf{k}_{i,j}$$

evaluated over the unit square, where:

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

is a Bernstein polynomial, and

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

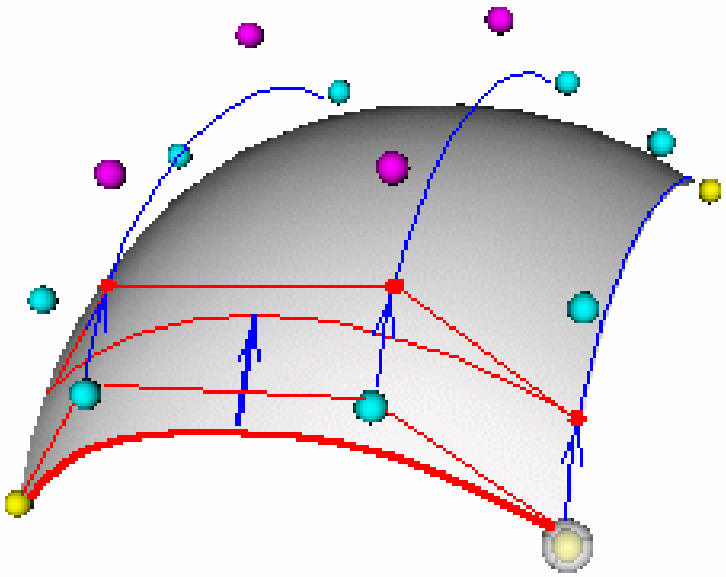
is the binomial coefficient.

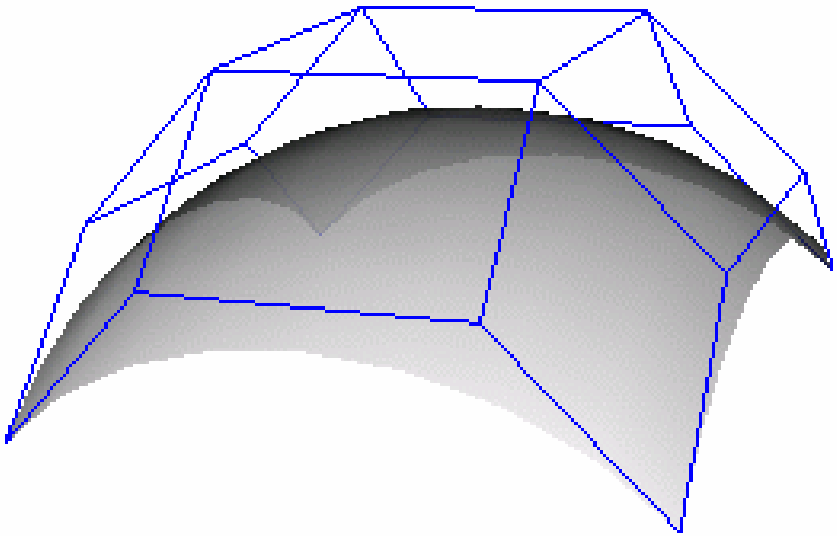
Geometry: Surfaces

Bézier

$$S(u,v) = \Sigma_{i=0,n}\Sigma_{j=0,m}\ B_i^n(u)\ B_j^m(v)\ \mathbf{P}_{ij}$$

$(n+1)(n+1)$ points $\mathbf{P}_{ij}$

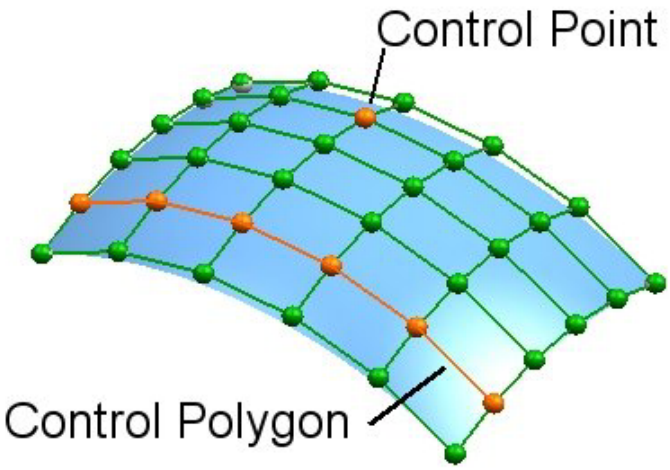


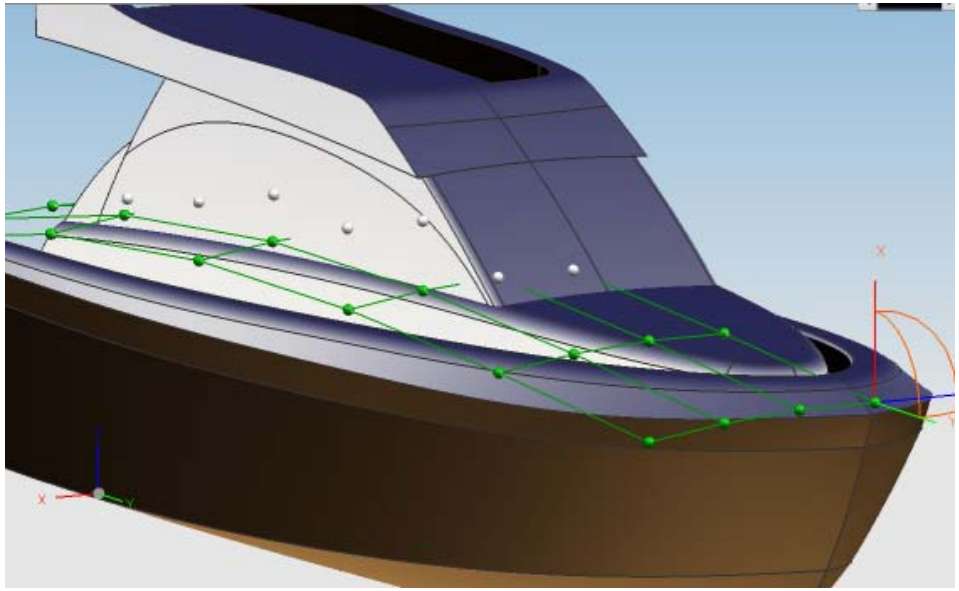


$$\mathbf{P}_i(v) = \Sigma_{j=0,m}\ B_j^m(v)\mathbf{P}_{ij}$$

Geometry: Surfaces

NURBS





NURBS are nearly ubiquitous for computer-aided design (CAD), manufacturing (CAM), and engineering (CAE)
and are part of numerous industry wide used standards,
such as IGES, STEP, ACIS, Parasolid.

NURBS: Properties

NURBS curves and surfaces are useful for a number of reasons:

- They are invariant under affine as well as perspective transformations.
- They offer one common mathematical form for both standard analytical shapes (e.g., conics) and free-form shapes.
- They provide the flexibility to design a large variety of shapes.
- They reduce the memory consumption when storing shapes (compared to simpler methods).
- They can be evaluated reasonably quickly by numerically stable and accurate algorithms.
- They are generalizations of non-rational B-Splines and non-rational and rational Bézier curves and surfaces.